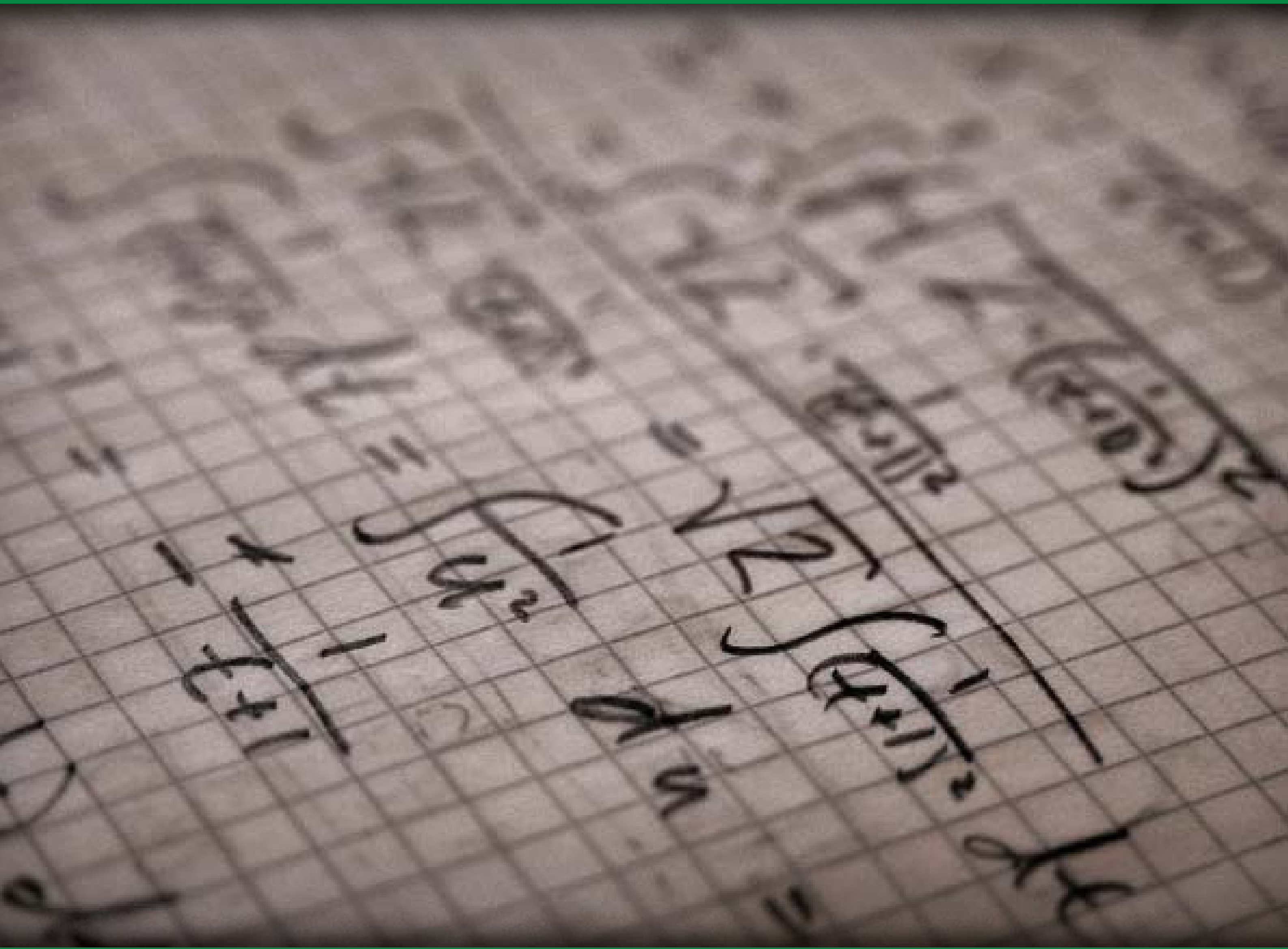
LINEAR ALGEBRA EXAMPLES C-4

QUADRATIC EQUATIONS IN TWO OR THREE VARIABLES

LEIF MEJLBRO



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Linear Algebra Examples c-4

Quadratic Equations in Two or Three Variables

Linear Algebra Examples c-4 – Quadratic equations in two or three variables

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Introduction

Here we collect all tables of contents of all the books on mathematics I have written so far for the publisher. In the rst list the topics are grouped according to their headlines, so the reader quickly can get an idea of where to search for a given topic.In order not to make the titles too long I have in the numbering added

a for a compendium

b for practical solution procedures (standard methods etc.)

c for examples.

The ideal situation would of course be that all major topics were supplied with all three forms of books, but this would be too much for a single man to write within a limited time.

After the rst short review follows a more detailed review of the contents of each book. Only Linear Algebra has been supplied with a short index. The plan in the future is also to make indices of every other book as well, possibly supplied by an index of all books. This cannot be done for obvious reasons during the rst couple of years, because this work is very big, indeed.

It is my hope that the present list can help the reader to navigate through this rather big collection of books.

Finally, since this list from time to time will be updated, one should always check when this introduction has been signed. If a mathematical topic is not on this list, it still could be published, so the reader should also check for possible new books, which have not been included in this list yet.

Unfortunately errors cannot be avoided in a rst edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
5th October 2008

1 Conic sections

Example 1.1 *Find the type and the position of the conic section, which is given by the equation*

$$x^2 + y^2 + 2x - 4y - 20 = 0.$$

First step. Elimination of the terms of first degree:

$$\begin{aligned} 0 &= x^2 + 2x + (1-1) + y^2 - 4y + (4-4) - 20 \\ &= (x+1)^2 + (y-2)^2 - 25. \end{aligned}$$

Second step. Rearrangement:

$$(x+1)^2 + (y-2)^2 = 25 = 5^2.$$

The conic section is a circle of centrum $(-1, 2)$ and radius 5.

Example 1.2 *Find the type and position of the conic section, which is given by the equation*

$$y^2 - 6y - 4x + 5 = 0.$$

We get by a small rearrangement,

$$y^2 - 6y + 9 = (y-3)^2 = 4x - 5 + 9 = 4(x+1).$$

The conic section is a parabola of vertex $(-1, 3)$, of horizontal axis of symmetry and $p = 4$, and with the focus

$$\left(x_0 + \frac{p}{4}, y_0\right) = (0, 3).$$

Example 1.3 *Find the type and position of the conic section, which is given by the equation*

$$3x^2 - 4y^2 + 12x + 8y - 4 = 0.$$

We first collect all the x and all the y separately:

$$\begin{aligned} 0 &= 3x^2 + 12x - 4y^2 + 8y - 4 \\ &= 3(x^2 + 4x + 4 - 4) - 4(y^2 - 2 + 1) \\ &= 3(x+2)^2 - 12 - 4(y-1)^2. \end{aligned}$$

Then by a rearrangement and by norming,

$$1 = \frac{1}{4}(x+2)^2 - \frac{1}{3}(y-1)^2 = \left(\frac{x+2}{2}\right)^2 - \left(\frac{y-1}{\sqrt{3}}\right)^2.$$

The conic section is an hyperbola of centrum $(-2, 1)$ and the half axes of the lengths $a = \frac{1}{2}$ and $b = \frac{1}{\sqrt{3}}$.

Example 1.4 *Find the type and position of the conic section, which is given by the equation*

$$x^2 + 5y^2 + 2x - 20y + 25 = 0.$$

It follows by a rearrangement,

$$\begin{aligned} 0 &= x^2 + 2x + (1-1) + 5(y^2 - 4y + 4 - 4) + 25 \\ &= (x+1)^2 + 5(y-2)^2 + 4. \end{aligned}$$

This conic section is the empty set, because the right hand side is ≥ 4 for every $(x, y) \in \mathbb{R}^2$.

Example 1.5 *Find the type and position of the conic section, which is given by the equation*

$$2x^2 + 3y^2 - 4x + 12y - 20 = 0.$$

It follows by a rearrangement that

$$\begin{aligned} 0 &= 2(x^2 - 2x + 1 - 1) + 3(y^2 + 4y + 4 - 4) - 20 \\ &= 2(x-1)^2 - 2 + 3(y+2)^2 - 12 - 20. \end{aligned}$$

Then by another rearrangement,

$$2(x-1)^2+3(y+2)^2=34,$$

hence by norming

$$\left(\frac{x-1}{\sqrt{17}}\right)^2+\left(\frac{y+2}{\sqrt{\frac{34}{3}}}\right)^2=1.$$

This conic section is an ellipse of centrum $(1,-2)$ and half axes

$$a=\sqrt{17} \qquad \text{and} \qquad b=\sqrt{\frac{34}{3}}.$$

Remark 1.1 This example clearly stems from the first half of the twentieth century. Apparently, a long time ago someone has made an error when copying the text, because the slightly changed formulation

$$2x^2+3y^2-4x+12y-22=0$$

would produce nicer results in the style of the past. No one has ever since made this correction. $\Diamond$

Example 1.6 *Prove that there is precisely one conic secion which goes through the following five points*

1. $(4,0)$, $(0,0)$, $(4,2)$, $\left(\frac{16}{3},\frac{4}{3}\right)$, $\left(\frac{4}{3},-\frac{2}{3}\right)$.
2. $(4,0)$, $(0,0)$, $(4,2)$, $\left(\frac{16}{3},\frac{4}{3}\right)$, $\left(\frac{4}{3},\frac{2}{3}\right)$.

Find the equation of the conic section and determine its type.

The general equation of a conic section is

$$Ax^2+By^2+2Cxy+2Dx+2Ey+F=0,$$

where A, ..., F are the six unknown constants. Then by insertion,

$$\begin{array}{rcrcrcrcrcrcl}
16A & & & & & + & 8D & & & + & F & = & 0,\\
 & & & & & & & & & & F & = & 0,\\
16A & + & 4B & + & 16C & + & 8D & + & 4E & + & F & = & 0,\\
\left(\frac{16}{3}\right)^2 A & + & \left(\frac{4}{3}\right)^2 B & + & 2\cdot\frac{16}{3}\cdot\frac{4}{3}C & + & \frac{32}{3}D & + & \frac{8}{3}E & + & F & = & 0,\\
\left(\frac{4}{3}\right)^2 A & + & \left(\frac{2}{3}\right)^2 B & \mp & \frac{16}{9}C & + & \frac{8}{3}D & \mp & \frac{4}{3}E & + & F & = & 0,
\end{array}$$

where $\mp$ is used with the upper sign corresponding to 1), and the lower sign corresponds to 2).

It follows immediately that $F = 0$ and $D = -2A$, hence the equations are reduced to

$$\begin{array}{rcrcrcrcl} & & 4B & + & 16C & + & 4E & = & 0, \\ \{16^2 - 192\}A & + & 16B & + & 128C & + & 24E & = & 0, \\ \{16 - 3 \cdot 16\}A & + & 4B & \mp & 16C & \mp & 12E & = & 0, \end{array}$$

and whence

$$\begin{array}{rcrcrcrcl} & & B & + & 4C & + & E & = & 0, \\ 8A & + & 2B & + & 16C & + & 3E & = & 0, \\ -8A & + & B & \mp & 4C & \mp & 3E & = & 0. \end{array}$$

1. In this case we get the equations $F = 0$, $D = -2A$ and

$$\left\{ \begin{array}{rcrcrcrcl} & & B & + & 4C & + & E & = & 0, \\ 8A & + & 2B & + & 16C & + & 3E & = & 0, \\ -8A & + & B & - & 4C & - & 3E & = & 0, \end{array} \right.$$

thus in particular,

$$\left\{ \begin{array}{rcrcrcl} B & + & 4C & + & E & = & 0, \\ 3B & + & 12C & & & = & 0, \end{array} \right.$$

and hence $E = 0$ and $B = -4C$. Then by insertion,

$$A = \frac{1}{8}(B - 4C - 3E) = \frac{1}{8}(-4C - 4C) = -C \quad \text{and} \quad D = -2A = 2C.$$

If we choose $A = 1$, then we get $C = -1$, $B = 4$, $D = -2$, $E = 0$, $F = 0$, and the equation becomes

$$x^2 + 4y^2 - 2xy - 4x = 0.$$

This is then written in the form

$$(x \ \ y) \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2(-2 \ \ 0) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad \text{where } \mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}.$$

Since

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 4) - 1 = \lambda^2 - 5\lambda + 3$$

has the roots

$$\lambda = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 3} = \frac{5}{2} \pm \sqrt{13},$$

where both roots are positive, the conic section is an ellipse.

2. In this case we get the equations $F = 0$, $D = -2A$ and

$$\left\{ \begin{array}{rcrcrcrcl} & & B & + & 4C & + & E & = & 0, \\ 8A & + & 2B & + & 16C & + & 3E & = & 0, \\ -8A & + & B & + & 4C & + & 3E & = & 0, \end{array} \right.$$

thus in particular,

$$\begin{cases} B + 4C + E = 0, \\ 3B + 20C + 6E = 0, \end{cases}$$

and hence $8C + 3E = 0$. If we choose $E = 8$, then $C = -3$ and

$$B = -4C - E = 4$$

and

$$A = \frac{1}{8}(B + 4C + 3E) = \frac{1}{8}(4 - 12 + 24) = 2,$$

and $D = -4$ and $F = 0$.

The conic section has the equation

$$2x^2 + 4y^2 - 6xy - 8x + 16y = 0.$$

The corresponding matrix is

$$\mathbf{A} = \begin{pmatrix} A & C \\ C & B \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}$$

where

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & -3 \\ -3 & 4-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 4) - 9 = \lambda^2 - 6\lambda - 1.$$

The roots of the characteristic polynomial are $\lambda = 3 \pm \sqrt{10}$, of which one is positive and the other one negative. We therefore conclude that the conic section i this case is an hyperbola.

Example 1.7 *Given in an ordinary rectangular coordinate system in the plane the points $A : (2, 0)$, $B : (-2, 0)$ and $C : (0, 4)$. Prove that there exists precisely one ellipse, which goes through the midpoints of the edges of triangle ABC, and in these points has the edges of the triangle as tangents.*

It follows by a geometric analysis that the three midpoints are

$(0, 0)$,	[horizontal tangent]
$(-1, 2)$,	[slope 2]
$(1, 2)$,	[slope -2].

We conclude from the symmetry that the half axes must be parallel to the coordinate axes for any possible ellipse which is a solution. Hence, the equation of the ellipse must necessarily be of the form

$$Ax^2 + By^2 + 2Dx + 2Ey + F = 0$$

without the product term $2Cxy$. Since we also have symmetry with respect to the y-axis, we must have $D = 0$, hence a possible equation must be of the structure

$$Ax^2 + By^2 + 2Ey + F = 0.$$

Furthermore, the ellipse goes through $(0,0)$, so $F = 0$.
Thus we have reduced the equation to

$$ax^2 + (y-b)^2 = b^2$$

with some new constants a, b.

If $(x,y) = (\pm 1, 2)$, then we get by insertion,

$$a + (2-b)^2 = b^2, \qquad \text{thus} \qquad a - 4b + 4 = 0.$$

If $y > b$, then the ellipse is the *graph* of

$$y = b + \sqrt{b^2 - ax^2},$$

thus

$$y' = -\frac{ax}{\sqrt{b^2 - ax^2}} \quad \text{where} \quad y'(-1) = 2 = \frac{a}{\sqrt{b^2 - a}},$$

hence

$$4b^2 - 4a = a^2, \qquad \text{and} \qquad 2b = \frac{a}{2} + 2,$$

and

$$4b^2 = a^2 + 4a = \frac{a^2}{4} + 2a + 4,$$

and we have $\frac{3}{4}\,a^2 + 2a - 4 = 0$, or put in another way

$$3a^2 + 8a - 16 = 0, \qquad \text{thus} \qquad a^2 + \frac{8}{3}\,a - \frac{16}{3} = 0.$$

From $a > 0$ follows that

$$a = -\frac{4}{3} - \sqrt{\frac{16}{9} + \frac{16}{3}} = -\frac{4}{3} + \frac{8}{3} = \frac{4}{3}.$$

Then

$$b = 1 + \frac{a}{4} = \frac{4}{3},$$

and the equation of the ellipse becomes

$$\frac{4}{3}\,x^2 + \left(y - \frac{4}{3}\right)^4 = \left(\frac{4}{3}\right)^2,$$

er put in a normed form,

$$\left(\frac{x}{2\sqrt{3}}\right)^2 + \left(\frac{y - \frac{4}{3}}{4/3}\right)^2 = 1.$$

Example 1.8 *Given in an ordinary rectangular coordinate system in the plane a conic section by the equation*

$$9x^2 + 16y^2 - 24xy - 40x - 30y + 250 = 0.$$

Find the type of the conic section, and show on a figure the position of the conic section with respect to the given coordinate system.

Here, $A = 9$, $B = 16$, $C = -12$, $D = -20$, $E = -15$ and $F = 250$, so it follows from a well-known formula that we can rewrite the equation in the form

$$(x \;\; y)\begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 2(-20 \quad -15)\begin{pmatrix} x \\ y \end{pmatrix} + 250 = 0.$$

The matrix

$$\mathbf{A} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix}$$

has the characteristic polynomial

$$\det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda-9)(\lambda-16)-144 = \lambda^2-25\lambda = \lambda(\lambda-25),$$

so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 25$.

If $\lambda_1 = 0$, then

$$\mathbf{A}-\lambda_1\mathbf{I} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \sim \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix},$$

hence an eigenvector is e.g. $\mathbf{v}_1 = (4,3)$. Then by norming,

$$\mathbf{q}_1 = \frac{1}{5}\,(4,3).$$

If $\lambda_2 = 25$, then

$$\mathbf{A}-\lambda_2\mathbf{I} = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix},$$

hence an eigenvector is e.g. $\mathbf{v}_2 = (-3,4)$. Then by norming,

$$\mathbf{q}_2 = \frac{1}{5}(-3,4).$$

We have now constructed the orthogonal substitution

$$\mathbf{Q} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{Q}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad \mathbf{Q}^{-1} = \mathbf{Q}^T,$$

hence by insertion,

$$\begin{aligned}
0 &= (x_1 \;\; y_1)\begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 2(-20 \quad -15)\,\frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 250 \\
&= 25y_1^2 + 2(-4 \quad -3)\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 250 \\
&= 25y_1^2 + 2(-25 \;\; 0)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 250 \\
&= 25\{y_1^2 - 2x_1 + 10\}.
\end{aligned}$$

This equation is reduced to the parabola

$$x_1 = \frac{1}{2}\,y_1^2 + 5, \qquad \text{possibly} \qquad x_1 - 5 = \frac{1}{2}\,y_1^2,$$

in the new coordinate system of vertex $(x_1, y_1) = (5,0)$ and the x_1-axis as its axis.

The transformation formulæ are

$$\begin{cases} x = \frac{4}{5}\,x_1 - \frac{3}{5}\,y_1, & x_1 = \frac{4}{5}\,x + \frac{3}{5}\,y, \\ y = \frac{3}{5}\,x_1 + \frac{4}{5}\,y_1, & y_1 = -\frac{3}{5}\,x + \frac{4}{5}\,y, \end{cases}$$

hence the vertex $(x_1, y_1) = (5, 0)$ corresponds to $(x, y) = (4, 3)$, and the x_1-axis corresponds to $y_1 = 0$, i.e. to the axis $y = \dfrac{3}{4}\,x$.

Example 1.9 *Describe for every a the type of the conic section*

$$(a+3)x^2 + 8xy + (a-3)y^2 + 10x - 20y - 45 = 0.$$

It follows by identification that

$$A = a+3, \quad B = a-3, \quad C = 4, \quad D = 5, \quad E = -10, \quad F = -45.$$

We first consider the matrix

$$\mathbf{A} = \begin{pmatrix} a+3 & 4 \\ 4 & a-3 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} (a-\lambda)+3 & 4 \\ 4 & (a-\lambda)-3 \end{vmatrix} = (a-\lambda)^2 - 9 - 16 = (\lambda - a)^2 - 25,$$

hence the eigenvalues are $\lambda = a \pm 5$.

If $|a| < 5$, then the roots have different signs, so we get hyperbolas or straight lines in this case.

If $|a| = 5$, we get parabolas, a straight line or the empty set.

If $|a| > 0$, then we get ellipses, a point or the empty set.

If $\lambda = a + 5$, then

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix},$$

thus an eigenvector is e.g. $(2, 1)$ of length $\sqrt{5}$.

If $\lambda = a - 5$, then

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix},$$

hence an eigenvector is e.g. $(-1, 2)$, of length $\sqrt{5}$.

The corresponding substitution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{Q} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad \text{hvor} \qquad \mathbf{Q} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

It follows from

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2x_1 - y_1 \\ x_1 + 2y_1 \end{pmatrix}$$

that

$$10x - 20y = \frac{10}{\sqrt{5}}\left(2x_1 - y_1 - 2x_1 - 4y_1\right) = -10\sqrt{5}\,y_1,$$

and the equation of the conic section is transformed into

$$(1) \qquad (a+5)x_1^2 + (a-5)y_1^2 - 10\sqrt{5}\,y_1 - 45 = 0.$$

If $a = 5$, this is reduced to $10x_1^2 - 10\sqrt{5}\,y_1 - 45 = 0$, i.e. to

$$y_1 = \frac{1}{\sqrt{5}}\,x_1^2 - \frac{9\sqrt{5}}{10}, \qquad a = 5,$$

which is the equation of a parabola .

If $a = -5$, then (1) is reduced to

$$-10y_1^2 - 10\sqrt{5}\,y_1 - 45 = 0,$$

thus to

$$0 = y_1^2 + \sqrt{5}\,y_1 + \frac{9}{2} = \left(y_1 + \frac{\sqrt{5}}{2}\right)^2 + \frac{9}{2} - \frac{5}{4} = \left(y_1 + \frac{\sqrt{5}}{2}\right)^2 + \frac{13}{4},$$

which has no solution, so we get the empty set for $a = -5$.

If $|a| < 5$, then (1.9) becomes

$$(a+5)x_1^2 - (5-a)\left\{y_1^2 + 2\cdot\frac{5\sqrt{5}}{5-a} + \frac{45}{5-a}\right\} = 0.$$

If the expression in $\{\cdots\}$ can be written as a square, we obtain two straight lines. Hence the requirements are $-5 < a < 5$ and

$$\frac{45}{5-a} = \left\{\frac{5\sqrt{5}}{5-a}\right\}^2 = \frac{125}{(5-a)^2}, \qquad \text{dvs.} \qquad 5-a = \frac{125}{45} = \frac{25}{9},$$

from which

$$a = 5 - \frac{25}{9} = \frac{20}{9} \in]-5,5[.$$

For this $a = \dfrac{20}{9}$ the conic section is degenerated into two straight lines.

If $a \in]-5,5[$ and $a \neq \dfrac{20}{9}$, then the conic section is an hyperbola.

If $a > 5$, it follows from (1) that

$$(a+5)x_1^2 + (a-5)\left\{y_1^2 - 2\cdot\frac{5\sqrt{5}}{a-5}\,y_1 + \frac{125}{(a-5)^2}\right\} = \frac{125}{a-5} + 45,$$

hence

$$(a+5)x_1^2 + (a-5)\left(y_1 - \frac{5\sqrt{5}}{a-5}\right)^2 = \frac{125}{a-5} + 45 > 0,$$

which describes an ellipse.

If $a < -5$, then it follows form (1) by a change of sign that

$$(-a-5)x_1^2 + (5-a)y_1^2 + 10\sqrt{5}\,y_1 + 45 = 0,$$

where $-a \pm 5 > 0$. We now have to have a closer look on the latter three terms. We see that

$$\begin{aligned}&(5-a)y_1^2 + 10\sqrt{5}\,y_1 + 45\\ &\qquad = (5-a)\left\{y_1^2 + 2\cdot\frac{5\sqrt{5}}{5-a}\,y_1 + \frac{45}{5-a}\right\}\\ &\qquad = (5-a)\left\{y_1 + \frac{5\sqrt{5}}{5-a}\right\}^2 - \frac{125}{5-a} + 45.\end{aligned}$$

Then notice that if $a < -5$, then $5-a > 10$, hence

$$45 - \frac{125}{5-a} > 45 - \frac{125}{10} = 45 - \frac{25}{2} > 0.$$

Thus, the equation of the conic section is a sum of three nonnegative terms, of which at least one is positive. Therefore, it can never be 0. We conclude that we do not have any solution, hence the set of solutions is empty in this case.

Summing up we get

$$\begin{array}{ll} a \le -5, & \text{no solution,} \\ \left\{ \begin{array}{l} -5 < a < 5, \\ a \ne \dfrac{20}{9}, \end{array} \right\} & \text{hyperbola ,} \\ a = \dfrac{20}{9}, & \text{two straight lines,} \\ a = 5, & \text{parabola} \\ a > 5, & \text{ellipse.} \end{array}$$

Example 1.10 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{pmatrix}.$$

Find a diagonal matrix $\mathbf{\Lambda}$ *and a proper orthogonal matrix* $\mathbf{Q}$*, such that* $\mathbf{\Lambda} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$.
Sketch the curve in an ordinary rectangular coordinate system in the plane of the equation

$$5x^2 + 7y^2 + 2\sqrt{3}\,xy = 1.$$

First find the characteristic polynomial

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 5)(\lambda - 7) - 3 = \lambda^2 - 12\lambda + 32 = (\lambda - 4)(\lambda - 8).$$

The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 8$. If $\lambda_1 = 4$, then

$$\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{pmatrix},$$

hence an eigenvector is e.g. $\mathbf{v}_1 = (\sqrt{3}, 1)$ of length 2. The corresponding normed vector is then

$$\mathbf{q}_1 = \frac{1}{2}(\sqrt{3}, 1).$$

If $\lambda_2 = 8$, then

$$\mathbf{A} - \lambda_2\mathbf{I} = \begin{pmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \sim \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix},$$

hence an eigenvector is e.g. $\mathbf{v}_2 = (-1, \sqrt{3})$ of length 2. The corresponding normed vector is

$$\mathbf{q}_2 = \frac{1}{2}(-1, \sqrt{3}).$$

We obtain the orthogonal matrix

$$\mathbf{Q} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}, \quad \text{svarende til} \quad \mathbf{\Lambda} = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}.$$

Finally, we get

$$\begin{aligned} 1 &= 5x^2 + 7y^2 + 2\sqrt{3}\,xy = (x \;\; y)\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x \;\; y)\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = (x_1 \;\; y_1)\mathbf{\Lambda}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \end{aligned}$$

where

$$4x_1^2 + 8y_1^2 = 1$$

is the equation of an ellipse.

Example 1.11 *Given in ordinary rectangular coordinates in the plane a conic section by the equation*

$$4x^2 + 11y^2 + 24xy + 40x - 30y - 105 = 0.$$

Describe the type and the position of the conic section.

From $A = 4$, $B = 11$ and $C = \frac{24}{2} = 12$ we get the matrix

$$\mathbf{A} = \begin{pmatrix} A & C \\ C & B \end{pmatrix} = \begin{pmatrix} 4 & 12 \\ 12 & 11 \end{pmatrix}$$

of the characteristic polynomial

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (\lambda - 4)(\lambda - 11) - 144 = \lambda^2 - 15\lambda - 100 \\ &= \lambda^2 - 2 \cdot \frac{15}{2}\,\lambda + \frac{225}{4} - \frac{225}{4} - 100 \\ &= \left(\lambda - \frac{15}{2}\right)^2 - \left(\frac{25}{2}\right)^2 = (\lambda - 20)(\lambda + 5), \end{aligned}$$

thus the eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = 20$.

If $\lambda_1 = -5$, then

$$\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \sim \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}.$$

A normed eigenvector is

$$\mathbf{q}_1 = \frac{1}{5}(4, -3).$$

If $\lambda_2 = 20$, then

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix} \sim \begin{pmatrix} -4 & 3 \\ 0 & 0 \end{pmatrix},$$

thus a normed eigenvector is

$$\mathbf{q}_2 = \frac{1}{5}(3,4).$$

Then

$$\mathbf{Q} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}, \qquad \text{hvor} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{Q}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$

hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{5}\left(4x_1 + 3y_1, -3x_1 + 4y_1\right).$$

We get

$$40x - 30y = 8\left\{4x_1 + 3y_1\right\} - 6\left\{-3x_1 + 4y_1\right\} = 50x_1,$$

thus by the transformation, the equation is transferred into

$$-5x_1^2 + 20y_1^2 + 50x_1 - 105 = 0,$$

hence

$$0 = x_1^2 - 10x_1 - 4y_1^2 + 21 = (x_1 - 5)^2 + 4y_1^2 - 4,$$

and we get an equation of an ellipse

$$(x_1 - 5)^2 + 4y_1^2 = 4, \quad \text{i.e.} \quad \left(\frac{x_1 - 5}{2}\right)^2 + y_1^2 = 1.$$

The centrum is $(x_1, y_1) = (5, 0)$, corresponding to $(x, y) = (4, -3)$. The half axes are $a = 2$ and $b = 1$.

It follows from

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \mathbf{Q}^T \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}(4x - 3y, 3x + 4y),$$

that the first half axis lies on $y_1 = 0$, i.e. on the line $3x + 4y = 0$, and the second half axis lies on $x_1 = 0$, i.e. on the line $y = \dfrac{4}{3}\,x$.

Example 1.12 *Given in an ordinary rectangular coordinate system in the plane a curve by the equation*

$$52x^2 + 73y^2 - 72xy - 200x - 150y + 525 = 0.$$

Describe the type and position of the curve, and find the parametric description of possible axes of symmetry.

It follows by identification that $A = 52$, $B = 73$ and $C = -36$, hence

$$\mathbf{A} = \begin{pmatrix} 52 & -36 \\ -36 & 73 \end{pmatrix}$$

and the characteristic polynomial is

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (\lambda - 52)(\lambda - 73) - 36^2 \\ &= \lambda^2 - 125\lambda + 52 \cdot 73 - 36^2 = \lambda^2 - 125\lambda + 2500 \\ &= \left(\lambda - \frac{125}{2}\right)^2 - \frac{5625}{4} = \left(\lambda - \frac{125}{2}\right)^2 - \left(\frac{75}{2}\right)^2 \\ &= (\lambda - 25)(\lambda - 100). \end{aligned}$$

The eigenvalues are $\lambda_1 = 25$ and $\lambda_2 = 100$.

If $\lambda_1 = 25$, then

$$\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} 27 & -36 \\ -36 & 48 \end{pmatrix} \sim \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix}.$$

A normed eigenvector is

$$\mathbf{q}_1 = \frac{1}{5}(4,3).$$

If $\lambda_2 = 100$, then

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} -48 & -36 \\ -36 & -27 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix}.$$

A normed eigenvector is

$$\mathbf{q}_2 = \frac{1}{5}(-3,4).$$

The orthogonal transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{Q} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad \text{hvor} \qquad \mathbf{Q} = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix},$$

hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{5}(4x_1 - 3y_1, 3x_1 + 4y_1)$$

and

$$\begin{aligned} -200x - 150y &= -40(4x_1 - 3y_1) - 30(3x_1 + 4y_1) \\ &= -160x_1 + 120y_1 - 90x_1 - 120y_1 = -250x_1. \end{aligned}$$

By this substitution the equation is transferred into

$$25x_1^2 + 100y_1^2 - 250x_1 + 525 = 0,$$

which we reduce to

$$0 = x_1^2 - 10x_1 + 4y_1^2 + 21 = (x_1 - 5)^2 + 4y_1^2 - 4,$$

i.e. to the equation of an ellipse

$$\left(\frac{x_1 - 5}{2} \right)^2 + y_1^2 = 1,$$

of centrum at $(x_1, y_1) = (5,0)$ and half axes $a_1 = 2$ (along the x_1-axis) and $b_1 = 1$ (along the y_1-axis).

It follows that the centrum is $(x,y) = (4,3)$ in the original coordinate system.

Since

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4x + 3y \\ -3x + 4y \end{pmatrix},$$

the direction of the x_1-axis is given by $y_1 = 0$, i.e. by the line $y = \frac{3}{4}x$, and the direction of the y_1-axis is given by $x_1 = 0$, i.e. by the line $y = -\frac{4}{3}x$.

Example 1.13 *Given in an ordinary rectangular coordinate system in the plane a point set M by the equation*

$$M : 4x^2 + 11y^2 + 24xy - 100y - 120 = 0.$$

1. *Prove that M is an hyperbola.*
2. *Find the coordinates of the centrum of M in the given coordinate system.*

1. Here, $A = 4$, $B = 11$ and $C = 12$, thus

$$\mathbf{A} = \begin{pmatrix} 4 & 12 \\ 12 & 11 \end{pmatrix}.$$

The characteristic polynomial

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 4-\lambda & 12 \\ 12 & 11-\lambda \end{vmatrix} = \lambda^2 - 15\lambda + 44 - 144 \\ &= \lambda^2 - 15\lambda - 100 = (\lambda+5)(\lambda-20) \end{aligned}$$

has the roots $\lambda_1 = -5$ and $\lambda_2 = 20$.

If $\lambda_1 = -5$, then

$$\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \sim \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix},$$

so a normed eigenvector is

$$\mathbf{q}_1 = \frac{1}{5}(4, -3).$$

If $\lambda_2 = 20$, then analogously

$$\mathbf{q}_2 = \frac{1}{5}(3, 4),$$

and the transformation is given by

$$\mathbf{Q} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}$$

where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{5}\left(4x_1 + 3y_1, -3x_1 + 4y_1\right).$$

In particular, the linear term is

$$-100y = -20(-3x_1 + 4y_1) = 60x_1 - 80y_1,$$

hence the equation of M is in the new coordinate system given by

$$-5x_1^2 + 20y_1^2 + 60x_1 - 80y_1 - 120 = 0,$$

which is reduced to

$$\begin{aligned} 0 &= x_1^2 - 4y_1^2 - 12x_1 + 16y_1 + 24 \\ &= \left\{x_1^2 - 12x_1 + 36\right\} - 36 - 4\left\{y_1^2 - 4y_1 + 4 - 4\right\} + 24 \\ &= (x_1 - 6)^2 - 4(y_1 - 2)^2 + 4. \end{aligned}$$

Then by a rearrangement and norming,

$$-\left(\frac{x_1 - 6}{2}\right)^2 + (y_1 - 2)^2 = 1.$$

2. The conic section is an hyperbola of centrum $(x_1, y_1) = (6, 2)$, corresponding to $(x, y) = (6, -2)$ in the old coordinate system.

   The half axes are $a = 2$ along the x_1-axis, i.e. the line $3x + 4y = 0$ in the old coordinate system, and $b = 1$ along the y_1-axis, i.e. parallel to the line $4x - 3y = 0$.

2 Conical surfaces

Example 2.1 *Find the type and position of the conical surface of the equation*

$$16y^2 - 9x^2 + 4z^2 - 36x - 64y - 24z = 80.$$

We have no product term, so the axes are parallel to the coordinate axes. We get by a rearrangement

$$\begin{aligned} 0 &= -9x^2 - 36x + 16y^2 - 64y + 4z^2 - 24z - 80 \\ &= -9\{x^2 - 4x + 4 - 4\} + 16\{y^2 - 4y + 4 - 4\} + 4\{z^2 - 6z + 9 - 9\} - 80 \\ &= -9(x-2)^2 + 16(y-2)^2 + 4(z-3)^2 + 36 - 64 - 36 - 80 \\ &= -9(x-2)^2 + 16(y-2)^2 + 4(z-3)^2 - 12^2, \end{aligned}$$

thus by a rearrangement and norming

$$-\left(\frac{x-2}{4}\right)^2 + \left(\frac{y-2}{3}\right)^2 + \left(\frac{z-3}{6}\right)^2 = 1.$$

This describes an hyperboloid of one 1 sheet and centrum $(2,2,3)$ and half axes $a = 4$, $b = 3$, $c = 6$.

Example 2.2 *Find the type and position of the conical surface of the equation*

$$2x^2 - y^2 - 3z^2 - 8x - 6y + 24z - 49 = 0.$$

We get by a rearrangement

$$\begin{aligned} 0 &= 2(x^2 - 4x + 4 - 4) - (y^2 + 6y + 9 - 9) - 3(z^2 - 8z + 16 - 16) - 49 \\ &= 2(x-2)^2 - (y+3)^2 - 3(z-4)^2 - 8 + 9 + 48 - 49 \\ &= 2(x-2)^2 - (y+3)^2 - 3(z-4)^2, \end{aligned}$$

thus

$$(x-2)^2 = \frac{1}{2}(y+3)^2 + \frac{3}{2}(z-4)^2.$$

This equation describes a second order cone of centrum $(2,-3,4)$.

Example 2.3 *Find the type and position of the conical surface of equation*

$$4y^2 - 3x^2 - 6z^2 - 16y - 6x + 36z - 77 = 0.$$

It follows by a rearrangement that

$$\begin{aligned} 0 &= -3x^2 - 6x + 4y^2 - 16y - 6z^2 + 36z - 77 \\ &= -3\{x^2 + 2x + 1 - 1\} + 4\{y^2 - 4y + 4 - 4\} - 6\{z^2 - 6z + 9 - 9\} - 77 \\ &= -3(x+1)^2 + 4(y-2)^2 - 6(z-3)^3 + 3 - 4 + 54 - 77 \\ &= -3(x+1)^2 + 4(y-2)^2 - 6(z-3)^2 - 24, \end{aligned}$$

hence

$$-\left(\frac{x+1}{2\sqrt{2}}\right)^2+\left(\frac{y-2}{\sqrt{6}}\right)^2-\left(\frac{z-3}{2}\right)^2=1.$$

This equation describes an hyperboloid of 2 sheets and half axes $a=2\sqrt{2}$, $b=\sqrt{6}$, $c=2$.

Example 2.4 *Find the type and position of the conical surface of the equation*

$$3z^2+5y^2-2x+10y-12z+21=0.$$

By a rearrangement,

$$\begin{aligned} 0 &= 5(y^2+2y+1-1)+3(z^2-4z+4-4)-2x+21 \\ &= 5(y+1)^2+3(z-2)^2-2x-5-12+21 \\ &= 5(y+1)^2+3(z-2)^2-2(x-2), \end{aligned}$$

i.e.

$$x-2=\frac{5}{2}(y+1)^2+\frac{3}{2}(z-2)^2,$$

which describes an elliptic paraboloid of vertex $(2,-1,2)$.

Example 2.5 *Find the type and position of the conical surface of the equation*

$$7y^2+x^2-2x-56y+113=0.$$

We get by a rearrangement

$$\begin{aligned} 0 &= x^2-2x+11+7\{y^2-8y+16-16\}+113 \\ &= (x-1)^2+7(y-4)^2-1-112+113 \\ &= (x-1)^2+7(y-4)^2. \end{aligned}$$

The only solution is the point $(1,4)$.

3 Rectilinear generators

Example 3.1 *Find the two systems of rectilinear generators of the following surface*

$$x^2 - z^2 - y^2 + 1 = 0.$$

We write this equation in the following form

$$(y^2 + z^2) - x^2 = 1,$$

which describes an hyperboloid of one sheet.

We get in the plane $z = 1$ the two lines $y = \pm x$, thus

$$\ell : (x, x, 1) \qquad \text{and} \qquad m : (x, -x, 1).$$

The systems of generators are obtained by rotating thes lines around the X-axis.

Example 3.2 *Find the two systems of rectilinear generator on the surface given by*

$$x^2 + yz - 1 = 0.$$

The corresponding matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

and its characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (1 - \lambda)(\lambda - \frac{1}{2})(\lambda + \frac{1}{2}).$$

If $\lambda_1 = 1$, then we of course get the normed eigenvector

$$\mathbf{q}_1 = (1, 0, 0).$$

If $\lambda_2 = \dfrac{1}{2}$, then

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

and a normed eigenvector is

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}(0, 1, 1).$$

If $\lambda_3 = -\dfrac{1}{2}$, then a normed eigenvector is

$$\mathbf{q}_3 = \frac{1}{\sqrt{2}}(0, -1, 1).$$

We get by the coordinate transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{1}{\sqrt{2}}(y_1 - z_1) \\ \frac{1}{\sqrt{2}}(y_1 + z_1) \end{pmatrix}$$

that

$$x_1^2 + \frac{1}{2}\,y_1^2 - \frac{1}{2}\,z_1^2 = 1,$$

i.e. an hyperboloid of 1 sheet.

The rotation axis is the z_1-axis. We get in the plane $x_1 = x = 1$ the lines $z_1 = \pm y_1$, thus in the original coordinate system,

$$\frac{1}{\sqrt{2}}(-y+z) = \pm\frac{1}{\sqrt{2}}(y+z),$$

or slightly nicer,

$$-y + z = \pm(y+z).$$

It follows that either $y=0$ or $z=0$, hence

$$\ell : (1,0,z) \qquad \text{and} \qquad m : (1,y,0).$$

Then these lines are rotated around the z_1-axis, i.e. around the line $y=-z$, by which we get the systems of generators.

Example 3.3 *Find the two systems of rectilinear generators on the surface given by*

$$9x^2+4y^2-36z^2=36.$$

By taking the norm we get

$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2-z^2=1.$$

The axis of rotation is the z-axis.

If $x=2$, then $z=\pm\frac{1}{3}\,y$. The two families of generators are obtained by rotating

$$\ell : (2,3z,z) \qquad \text{and} \qquad m : (2,-3z,z)$$

around the z-axis.

Example 3.4 *Find the two systems of rectilinear generators on the surface given by the equation*

$$y^2-4z^2+2x=0.$$

This equation describes an hyperbolic paraboloid,

$$2z^2-\frac{1}{2}\,y^2=x$$

with the X-axis as symmetry axis. It follows from

$$\left(\sqrt{2}\,z+\frac{1}{\sqrt{2}}\,y\right)\left(\sqrt{2}\,z-\frac{1}{\sqrt{2}}\,y\right)=x$$

that the generators are

$$\left\{\begin{array}{l}\sqrt{2}\,z+\dfrac{1}{\sqrt{2}}\,y=k,\\ k\sqrt{2}\,z-\dfrac{k}{\sqrt{2}}\,y-x=0,\end{array}\right. \qquad \text{dvs.} \qquad \left\{\begin{array}{l}2z+y=2k_1,\\ 2z-y=\dfrac{1}{k_1}\,x,\end{array}\right.$$

where $k_1=k\sqrt{2}$, and analogously for the other family of lines,

$$\left\{\begin{array}{l}\sqrt{2}\,z-\dfrac{1}{\sqrt{2}}\,y=k,\\ k\sqrt{2}\,z+\dfrac{k}{\sqrt{2}}\,y-x=0,\end{array}\right. \qquad \text{dvs.} \qquad \left\{\begin{array}{l}2z-y=2k_1,\\ 2z+y=\dfrac{1}{k_1}\,x.\end{array}\right.$$

Example 3.5 *Find the two systems of rectilinear generators on the surface given by the equation:*

$$x^2 - 4y^2 = 4z.$$

We have again an hyperbolic paraboloid,

$$4z^2 - y^2 = (2z-y)(2z+y) = 2x,$$

hence the lines are, expressed by the parameter k,

$$\begin{cases} 2z+y=2k, \\ 2z-y=\dfrac{1}{k}\,x, \end{cases} \qquad \text{or} \qquad \begin{cases} 2z-y=2k, \\ 2z+y=\dfrac{1}{k}\,x. \end{cases}$$

Example 3.6 *Find the two systems of rectilinear generators on the surface of the equation:*

$$x^2 + y^2 - yz - 1 = 0.$$

We first write the equation in the following way

$$1 = x^2 + y^2 - yz + \frac{1}{4}\,z^2 - \left(\frac{z}{2}\right)^2 = x^2 + \left(y - \frac{1}{2}\,z\right)^2 - \left(\frac{z}{2}\right)^2,$$

which of course also can be found in the usual (and more tedious) way by first setting up the matrix, then find the eigenvalues and eigenvectors, etc..

The equation describes an hyperboloid of 1 sheet and with the Z-axis as axis of rotation.

We get in the plane $x = 1$,

$$\left(y - \frac{z}{2}\right)^2 = \left(\frac{z}{2}\right)^2,$$

thus either $y = 0$ or $y = z$. The systems of generators are obtained by rotating

$$\ell : (1,0,z) \qquad \text{and} \qquad m : (1,z,z)$$

around the Z-axis.

4 Various surfaces

Example 4.1 *Given the ellipsoid*

$$Ax^2 + By^2 + Cz^2 = 1,$$

where $A < B < C$. Prove that there are two planes through the Y-axis, which both cut the ellipsoid in circles. Then prove that only planes parallel with one of these two planes will cut the ellipsoid in circles.

Whenever we are talking about an ellipsoid, we must also require that $A > 0$.

Any plane containing the Y-axis must have an equation of the form

$$ax + bz = 0, \qquad \text{where} \qquad (a, b) \neq (0, 0).$$

If $b \neq 0$, then $z = -\dfrac{a}{b}\,x$, hence by insertion

$$Ax^2 + By^2 + C\left(\frac{a}{b}\right)^2 x^2 = 1,$$

whence

$$\left\{A + C\left(\frac{a}{b}\right)^2\right\} x^2 + By^2 = 1.$$

We shall get a circle, when

$$A + C\left(\frac{a}{b}\right)^2 = B, \qquad \text{i.e. when} \qquad \frac{a}{b} = \pm\sqrt{\frac{B-A}{C}},$$

which precisely gives us two solutions, because $B > A$.

We notice that $b = 0$ gives $x = 0$, and since $B < C$, we only obtain one ellipse $By^2 + Cz^2 = 1$.

A plane parallel to the Y-axis has the equation

$$x = c \qquad \text{eller} \qquad z = ax + b.$$

Again $x = c$ will only give an ellipse. When we put $z = ax + b$, we get

$$\begin{aligned} 1 &= Ax^2 + By^2 + C(a^2x^2 + 2abx + b^2) \\ &= (A + a^2C)x^2 + 2abCx + b^2C + By^2. \end{aligned}$$

We only obtain a circle, if $A + a^2C = B$, so

$$a = \pm\sqrt{\frac{B-A}{C}}.$$

There must of course also be a constraint on b for given a. There is, however, some very good computational reasons for not to ask for this constraint.

A plan which is not parallel to the Y-axis must have the equation

$$y = ax + bz + c.$$

Without some additional knowledge of geometry, which cannot be assumed, this case becomes very difficult to describe.

Example 4.2 *Given in an ordinary rectangular coordinate system in space the surface of the equation*

$$x^2 + ay^2 + z^2 = 1.$$

Find all a, for which the surface contains at least one straight line, and find for each of them a parametric description of a straight line which lies on this surface.

If $a > 0$, the surface is an ellipsoid, thus there are no straight lines on the surface.

If $a = 0$, the surface is a cylindric surface. A parametric description of a straight line on the surface is

$$(x, y, z) = (\cos\varphi, y, \sin\varphi), \qquad y \in \mathbb{R},$$

where we for each fixed φ get a straight line.

If $a = -b^2 < 0$, the surface is an hyperboloid of 1 sheet. In the plane $x = 1$ we find the two straight lines $z = \pm by$, hence

$$\ell : (1, y, \sqrt{-a}\, y) \qquad \text{and} \qquad m : (1, y, -\sqrt{-a}\, y), \qquad y \in \mathbb{R},$$

are two lines on the surface. We get all such straight lines by rotating these around the Y-axis.

Example 4.3 *Given in an ordinary rectangular coordinate system XYZ of positive orientation in space a conical surface of the equation*

$$50x^2 + 25y^2 + az^2 - 100x - 200y - 2az = 0,$$

where $a \in \mathbb{R}$.
Find the type of conical surface, which is described by $a = 0$, $a = -250$, $a = -450$, respectively.

First write

$$\begin{aligned} 0 &= 50x^2 - 100x + 25y^2 - 200y + az^2 - 2az \\ &= 50\{x^2 - 2x + 1 - 1\} + 25\{y^2 - 8y + 16 - 16\} + a\{z^2 - 2z + 1 - 1\} \\ &= 50(x-1)^2 + 25(y-4)^2 + a(z-1)^2 - 50 - 400 - a, \end{aligned}$$

i.e.

$$(2) \qquad 50(x-1)^2 + 25(y-4)^2 + a(z-1)^2 = 450 + a.$$

1. If $a = 0$, then by norming

   $$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y-4}{3\sqrt{2}}\right)^2 + 0 \cdot (z-1)^2 = 1,$$

   which is the equation of an elliptic cylindric surface.

2. If $a = -250$, then

   $$50(x-1)^2 + 25(y-4)^2 - 250(z-1)^2 = 200,$$

   hence by norming,

   $$\left(\frac{x-1}{2}\right)^2 + \left(\frac{y-4}{2\sqrt{2}}\right)^2 - \left(\frac{z-1}{2/\sqrt{5}}\right)^2 = 1.$$

   This is the equation of an hyperboloid of 1 sheet.

3. If $a = -450$, then

   $$50(x-1)^2 + 25(y-4)^2 - 450(z-1)^2 = 0,$$

   hence

   $$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y-4}{3\sqrt{2}}\right)^2 - (z-1)^2 = 0.$$

   This is the equation of a conical surface.

Remark 4.1 It is not difficult to discuss (2) for general $a \in \mathbb{R}$. There are only two critical point, namely $a = 0$, where the coefficient of the quadratic term $(z-1)^2$ becomes 0, and $a = -450$, where the right hand side becomes 0. $\Diamond$

We get

$$\begin{array}{ll}
a > 0, & \text{ellipsoid,} \\
a = 0, & \text{elliptic cylindric surface,} \\
-450 < a < 0, & \text{hyperboloid of 1 sheet,} \\
a = -450, & \text{conical surface,} \\
a < -450, & \text{hyperboloid of 2 sheets.}
\end{array}$$

$\Diamond$

5 Conical surfaces

Example 5.1 *Find the type and position of the conical surface which is described by the equation*

$$z = xy.$$

By the change of variables (a rotation of $\frac{\pi}{4}$)

$$x = \frac{1}{\sqrt{2}}(-x_1 + y_1), \qquad y = \frac{1}{\sqrt{2}}(x_1 + y_1)$$

We get

$$z = xy = \frac{1}{2}\left(-x_1^2 + y_1^2\right),$$

and the surface is an hyperbolic paraboloid.

ALTERNATIVELY and more difficult we setup the corresponding matrix and then find the eigenvalues and eigenvectors. Of course, we end up with precisely the same transformation. ◊

Example 5.2 *Find the type and position of the conical surface, which is described by the equation*

$$z^2 = 2xy.$$

Applying the same rotation as in EXAMPLE 5.1 we get

$$z^2 = -x_1^2 + y_1^2,$$

hence

$$x_1^2 - y_1^2 + z^2 = 0, \qquad \text{possibly} \qquad y_1^2 = x_1^2 + z^2,$$

which is the equation of a conical surface.

Example 5.3 *Find the type and position of the conical surface, which is described by the equation*

$$x^2 + y^2 - z^2 + 2xy - 2x - 4y - 1 = 0.$$

We get by a rearrangement,

$$\begin{aligned} 0 &= (x+y)^2 - z^2 - 3(x+y) + (x-y) - 1 \\ &= \left(x + y - \frac{3}{2}\right)^2 - z^2 + (x-y) - 1 - \frac{9}{4}, \end{aligned}$$

thus

$$x - y - \frac{13}{4} = z^2 - \left(x + y - \frac{3}{2}\right)^2,$$

hence by norming,

$$\frac{1}{\sqrt{2}}\,(x-y)-\frac{13}{4\sqrt{2}}=\frac{1}{\sqrt{2}}\,z^2-\sqrt{2}\left(\frac{1}{\sqrt{2}}(x+y)-\frac{3}{2\sqrt{2}}\right)^2.$$

If we put

$$x_1=\frac{1}{\sqrt{2}}(x+y)\qquad\text{and}\qquad y_1=\frac{1}{\sqrt{2}}(x-y),$$

[a rotation], it follows that

$$y_1-\frac{13}{4\sqrt{2}}=\frac{1}{\sqrt{2}}\,z^2-\sqrt{2}\left(x_1-\frac{3}{2\sqrt{2}}\right)^2,$$

which corresponds to a (rotated) hyperbolic paraboloid.

Example 5.4 *Find the type and position of the conical surface, which is described by the equation*

$$8x^2+11y^2+8z^2+4yz+8zx-4xy-16x+4y-8z-4=0.$$

We collect the quadratic terms in the matrix $\mathbf{A}$, i.e.

$$(x\;\;y\;\;z)\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix},\qquad\text{where}\qquad \mathbf{A}=\begin{pmatrix}8&-2&4\\-2&11&2\\4&2&8\end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix}8-\lambda&-2&4\\-2&11-\lambda&2\\4&2&8-\lambda\end{vmatrix}=\begin{vmatrix}12-\lambda&0&12-\lambda\\-2&11-\lambda&2\\4&2&8-\lambda\end{vmatrix}\\
&= (12-\lambda)\begin{vmatrix}1&0&1\\-2&11-\lambda&2\\4&2&8-\lambda\end{vmatrix}=(12-\lambda)\begin{vmatrix}1&0&0\\-2&11-\lambda&4\\4&2&4-\lambda\end{vmatrix}\\
&= (12-\lambda)\begin{vmatrix}11-\lambda&-4\\2&4-\lambda\end{vmatrix}=-(\lambda-12)\{\lambda^2-15\lambda+44-8\}\\
&= -(\lambda-12)(\lambda-3)(\lambda-12)=-(\lambda-2)(\lambda-12)^2.
\end{aligned}$$

If $\lambda_1=3$, then

$$\begin{aligned}
\mathbf{A}-\lambda_1\mathbf{I} &= \begin{pmatrix}5&-2&4\\-2&8&2\\4&2&5\end{pmatrix}\sim\begin{pmatrix}1&-4&-1\\-1&4&1\\0&18&9\end{pmatrix}\\
&\sim \begin{pmatrix}1&-4&-1\\0&2&1\\0&0&0\end{pmatrix}\sim\begin{pmatrix}1&0&1\\0&2&1\\0&0&0\end{pmatrix}.
\end{aligned}$$

An eigenvector is e.g. $\mathbf{v}_1 = (2,1,-2)$ of the length $\|\mathbf{v}_1\| = \sqrt{9} = 3$, thus a normed eigenvector is

$$\mathbf{q}_1 = \frac{1}{3}\,(2,1,-2).$$

If $\lambda_2 = \lambda_3 = 12$, then

$$\mathbf{A} - \lambda_2\mathbf{I} = \begin{pmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Two obvious linearly independent eigenvectors are given by

$$\mathbf{v}_2 = (1,0,1) \qquad \text{and} \qquad \mathbf{v}_3 = (0,2,1),$$

where $\|\mathbf{v}_2\| = \sqrt{2}$ and (by Gram-Schmidt)

$$\mathbf{v}_3 - \frac{1}{\|\mathbf{v}_2\|^2}(\mathbf{v}_3 \cdot \mathbf{v}_2)\,\mathbf{v}_2 = (0,2,1) - \frac{1}{2}\cdot 1 \cdot (1,0,1) = \frac{1}{2}(-1,4,1).$$

Here, $\|(-1,4,1)\| = \sqrt{1+16+1} = 3\sqrt{2}$, hence

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}(1,0,1) \qquad \text{and} \qquad \mathbf{q}_3 = \frac{1}{3\sqrt{2}}(-1,4,1).$$

The transformation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{Q}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \qquad \text{hvor} \qquad \mathbf{Q} = \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{pmatrix}.$$

Since

$$\begin{array}{rclclcl} x &=& \frac{2}{3}\,x_1 &=& \frac{1}{\sqrt{2}}\,y_1 &-& \frac{1}{3\sqrt{2}}\,z_1, \\ y &=& \frac{1}{3}\,x_1 && &+& \frac{4}{3\sqrt{2}}\,z_1, \\ z &=& -\frac{2}{3}\,x_1 &+& \frac{1}{\sqrt{2}}\,y_1 &+& \frac{1}{3\sqrt{2}}\,z_1, \end{array}$$

we get for the linear terms that

$$\begin{aligned} &-16x + 4y - 8z \\ &\quad = \left(-\frac{32}{3}+\frac{4}{3}+\frac{16}{3}\right)x_1 + \left(-\frac{16}{\sqrt{2}}-\frac{8}{\sqrt{2}}\right)y_1 + \left(\frac{16}{3\sqrt{2}}+\frac{16}{3\sqrt{2}}-\frac{8}{3\sqrt{2}}\right)z_1 \\ &\quad = -4x_1 - 12\sqrt{2}\,y_1 + 4\sqrt{2}\,z_1, \end{aligned}$$

and the equation is transferred into

$$\begin{aligned} 0 &= 3x_1^2 + 12y_1^2 + 12z_1^2 - 4x_1 - 12\sqrt{2}\,y_1 + 4\sqrt{2}\,z_1 - 4 \\ &= 3\left\{x_1^2 - \frac{4}{3}\,x_1 + \frac{4}{9} - \frac{4}{9}\right\} + 12\left\{y_1^2 - \sqrt{2}\,y_1 + \frac{1}{2} - \frac{1}{2}\right\} + 12\left\{z_1^2 + \frac{\sqrt{2}}{3}\,z_1 + \frac{1}{18} - \frac{1}{18}\right\} - 4 \\ &= 3\left(x_1 - \frac{2}{3}\right)^2 + 12\left(y_1 - \frac{1}{\sqrt{2}}\right)^2 + 12\left(z_1 - \frac{\sqrt{2}}{6}\right)^2 - \frac{4}{3} - 6 - \frac{2}{3} - 4. \end{aligned}$$

It follows from $-\frac{4}{3} - 6 - \frac{2}{3} - 4 = -12$ that the equation can be written

$$\frac{1}{2^2}\left(x_1 - \frac{2}{3}\right)^2 + \left(y_1 - \frac{1}{\sqrt{2}}\right)^2 + \left(z_1 - \frac{\sqrt{2}}{6}\right)^1 = 1,$$

thus the surface is an ellipsoid of centrum

$$(x_1, y_1, z_1) = \left(\frac{2}{3}, \frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{6}\right)$$

and half axes $a = 2$, $b = c = 1$.

Example 5.5 *Find the type and position of the conical surface, which is given by the equation*

$$x^2+4y^2+z^2+20yz+26zx+20xy-24=0.$$

The corresponding matrix is

$$\mathbf{A}=\begin{pmatrix} 1 & 10 & 13 \\ 10 & 4 & 10 \\ 13 & 10 & 1 \end{pmatrix}$$

of the characteristic polynomial

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} 1-\lambda & 10 & 13 \\ 10 & 4-\lambda & 10 \\ 13 & 10 & 1-\lambda \end{vmatrix}=\begin{vmatrix} 24-\lambda & 24-\lambda & 24-\lambda \\ 10 & 4-\lambda & 10 \\ 13 & 10 & 1-\lambda \end{vmatrix}$$

$$=-(\lambda-24)\begin{vmatrix} 1 & 1 & 1 \\ 10 & 4-\lambda & 10 \\ 13 & 10 & 1-\lambda \end{vmatrix}=-(\lambda-24)\begin{vmatrix} 1 & 0 & 0 \\ 10 & -6-\lambda & 0 \\ 13 & -3 & -12-\lambda \end{vmatrix}$$

$$=-(\lambda-24)(\lambda+6)(\lambda+12).$$

The three eigenvalues are $\lambda_1=-12$, $\lambda_2=-6$ and $\lambda_3=24$.

It $\lambda_1=-12$, then

$$\mathbf{A}-\lambda_1\mathbf{I}=\begin{pmatrix} 13 & 10 & 13 \\ 10 & 16 & 10 \\ 13 & 10 & 13 \end{pmatrix}\sim\begin{pmatrix} 3 & -6 & 3 \\ 5 & 8 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim\begin{pmatrix} 1 & -2 & 1 \\ 2 & 14 & 2 \\ 0 & 0 & 0 \end{pmatrix}\sim\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\sim\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_1=(1,0,-1)$, which is of length $\sqrt{2}$, hence a normed eigenvector is given by

$$\mathbf{q}_1=\frac{1}{\sqrt{2}}(1,0,-1).$$

If $\lambda_2=-6$, then

$$\mathbf{A}-\lambda_2\mathbf{I}=\begin{pmatrix} 7 & 10 & 13 \\ 10 & 10 & 10 \\ 13 & 10 & 7 \end{pmatrix}\sim\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\sim\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $(1,-2,1)$ of length $\sqrt{6}$, hence the corresponding normed eigenvector is

$$\mathbf{q}_2=\frac{1}{\sqrt{6}}(1,-2,1).$$

If $\lambda_3 = 24$, then

$$\begin{aligned}\mathbf{A}-\lambda_3\mathbf{I} &= \begin{pmatrix} -23 & 10 & 13 \\ 10 & -20 & 10 \\ 13 & 10 & -23 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 3 & 30 & -33 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 1 & 10 & -11 \\ 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 12 & -12 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

An eigenvector is e.g. $\mathbf{v}_3 = (1,1,1)$ of length $\sqrt{3}$, hence a normed eigenvector is

$$\mathbf{q}_3 = \frac{1}{\sqrt{3}}\,(1,1,1).$$

The coordinate transformation is then given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{Q}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \qquad \text{hvor} \qquad \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

We obtain by this transformation the equation

$$-12x_1^2 - 6y_1^2 + 24z_1^2 = 24,$$

thus by norming

$$-\frac{1}{2}\,x_1^2 - \frac{1}{4}\,y_1^2 + z_1^2 = 1.$$

This equation describes an hyperboloid of 2 sheets.

Example 5.6 *Find the type and position of the conical surface, which is given by the equation*

$$3x^2 + 3y^2 - 5z^2 - 8xy + 5 = 0.$$

If we only consider the variables (x,y), we get the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$$

of the characteristic polynomial $(\lambda-3)^3 - 4^2$, the roots of which are $\lambda_1 = -1$ and $\lambda_2 = 7$.

If $\lambda_1 = -1$, then

$$\mathbf{A}-\lambda_1\mathbf{I} = \begin{pmatrix} 4 & -4 \\ -44 & \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

A normed eigenvector is $\mathbf{q}_1 = \frac{1}{\sqrt{2}}\,(1,1,0)$. Analogously, $\mathbf{q}_2 = \frac{1}{\sqrt{2}}\,(1,-1,0)$ is a normed eigenvector for $\lambda_2 = 7$.

By the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

the equation is carried over into

$$-x_1^2 + 7y_1^2 - 5z_1^2 + 5 = 0,$$

hence by a rearrangement,

$$\frac{1}{5}\,x_1^2 - \frac{7}{5}\,y_1^2 + z_1^2 = 1.$$

This is the equation of an hyperboloid of sheet.

Example 5.7 *Find the type and position of the conical surface, which is given by the equation*

$$5x^2 + 8y^2 + 5z^2 - 4yz + 8zx + 4xy - 4x + 2y + 4z = 0.$$

The quadratic terms are represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 5 \end{pmatrix},$$

of the characteristic polynomial

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 5-\lambda & 2 & 4 \\ 2 & 8-\lambda & -2 \\ 4 & -2 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 9-\lambda & 0 & 9-\lambda \\ 2 & 8-\lambda & -2 \\ 4 & -2 & 5-\lambda \end{vmatrix} \\ &= -(\lambda-9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & 8-\lambda & -4 \\ 4 & -2 & 1-\lambda \end{vmatrix} = -(\lambda-9)\{\lambda^2 - 9\lambda + 8 - 8\} \\ &= -\lambda(\lambda-9)^2. \end{aligned}$$

If $\lambda_1 = 0$, then

$$\begin{aligned} \mathbf{A} - \lambda_1\mathbf{I} &= \begin{pmatrix} 5 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -1 \\ 1 & -14 & 8 \\ 0 & -18 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -1 \\ 0 & -18 & 9 \\ 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 4 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

An eigenvector is e.g. $\mathbf{v}_1 = (2, -1, -2)$ of length 3, hence a normed eigenvector is

$$\mathbf{q}_1 = \frac{1}{3}\,(2, -1, -2).$$

If $\lambda_2 = 9$, then

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} -4 & 2 & 4 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Two linearly independent eigenvectors are

$$\mathbf{v}_2 = (1,0,1) \qquad \text{and} \qquad \mathbf{v}_3 = (0,2,-1),$$

where $\|\mathbf{v}_2\| = \sqrt{2}$ and (by Gram-Schmidt)

$$\mathbf{v}_3 = \frac{1}{\|\mathbf{v}_2\|^2}(\mathbf{v}_3 \cdot \mathbf{v}_2)\,\mathbf{v}_2 = (0,2,-1) - \frac{1}{2}(-1)\cdot(1,0,1) = \frac{1}{2}\,(1,4,-1).$$

Since $\|(1,4,-1)\| = \sqrt{18} = 3\sqrt{2}$, the orthonormed eigenvectors are

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}\,(1,0,1) \qquad \text{and} \qquad \mathbf{q}_3 = \frac{1}{3\sqrt{2}}\,(1,4,-1).$$

The transformation is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}\,x_1 & + & \frac{1}{\sqrt{2}}\,y_1 & + & \frac{1}{3\sqrt{2}}\,z_1 \\ -\frac{1}{3}\,x_1 & & & + & \frac{4}{3\sqrt{2}}\,z_1 \\ -\frac{2}{3}\,x_1 & + & \frac{1}{\sqrt{2}}\,y_1 & - & \frac{1}{3\sqrt{2}}\,z_1 \end{pmatrix}.$$

Thus, the linear terms are transformed into

$$-4x+2y+4z = \left(-\frac{8}{3}-\frac{2}{3}-\frac{8}{3}\right)x_1 - \left(\frac{4}{\sqrt{2}}+\frac{4}{\sqrt{2}}\right)y_1 + \left(-\frac{4}{3\sqrt{2}}+\frac{8}{3\sqrt{2}}-\frac{4}{3\sqrt{2}}\right)z_1 = -6x_1.$$

We get by insertion that the equation is transformed into

$$0\cdot x_1^2 + 9y_1^2 + 9z_1^2 - 6x_1 = 0,$$

which can be written

$$x_1 = \frac{3}{2}(y_1^2+z_1^2).$$

This is the equation of an elliptic paraboloid.

Example 5.8 *Find the type and position of the conical surface, which is given by the equation*

$$5x^2 - 2y^2 + 11z^2 + 12xy + 12yz - 16 = 0.$$

Here we only have the constant and the terms of second order corresponding to the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 6 & 0 \\ 6 & -2 & 6 \\ 0 & 6 & 11 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} \det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} 5-\lambda & 6 & 0 \\ 6 & -2-\lambda & 6 \\ 0 & 6 & 11-\lambda \end{vmatrix} \\ &= (5-\lambda)\{\lambda^2-9\lambda-22-36\} - 36(11-\lambda) \\ &= (5-\lambda)(\lambda^2-9\lambda-58) - 396 + 36\lambda \\ &= -\lambda^3+9\lambda^2+58\lambda+5\lambda^2-45\lambda-290-396+36\lambda \\ &= -\lambda^3+14\lambda^2+49\lambda-686 = -\{\lambda^3-14\lambda-49\lambda+686\} \\ &= -(\lambda-7)(\lambda^2-7\lambda-98) = -(\lambda-7)(\lambda-14)(\lambda+7). \end{aligned}$$

If $\lambda_1 = -7$, then

$$\begin{aligned} \mathbf{A}-\lambda_1\mathbf{I} &= \begin{pmatrix} 12 & 6 & 0 \\ 6 & 5 & 6 \\ 0 & 6 & 18 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 6 & 5 & 6 \\ 0 & 1 & 3 \end{pmatrix} \\ &\sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

An eigenvector is e.g. $(3,-6,2)$ of length $\sqrt{9+36+4}=7$, thus a normed eigenvector is given by

$$\mathbf{q}_1 = \frac{1}{7}\,(3,-6,2).$$

If $\lambda_2 = 7$, then

$$\mathbf{A}-\lambda_2\mathbf{I} = \begin{pmatrix} -2 & 6 & 0 \\ 6 & -9 & 6 \\ 0 & 6 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 0 \\ 2 & -3 & 2 \\ 0 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $(6,2,-3)$ of length 7, hence a normed eigenvector is

$$\mathbf{q}_2 = \frac{1}{7}\,(6,2,-3).$$

If $\lambda_3 = 14$, then

$$\mathbf{A}-\lambda_3\mathbf{I} = \begin{pmatrix} -9 & 6 & 0 \\ 6 & -16 & 6 \\ 0 & 6 & -3 \end{pmatrix} \sim \begin{pmatrix} 3 & -2 & 0 \\ 3 & -8 & 3 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is $(2,3,6)$ of length 7, hence a normed eigenvector is

$$\mathbf{q}_3 = \frac{1}{7}\,(2,3,6).$$

The transformation matrix is

$$\mathbf{Q} = \frac{1}{7}\begin{pmatrix} 3 & 6 & 2 \\ -6 & 2 & 3 \\ 2 & -3 & 6 \end{pmatrix},$$

and the equation is by this carried into

$$0 = -7x_1^2 + 7y_1^2 + 14z_1^2 - 16,$$

hence, by a rearrangement,

$$-\frac{7}{16}\,x_1^2 + \frac{7}{16}\,y_1^2 + \frac{7}{8}\,z_1^2 = 1.$$

This is the equation of an hyperboloid of 1 sheet.

Example 5.9 *Find the type and position of the conical surface, which is given by the equation*

$$x^2 + yz = 0.$$

By the rotation

$$x = x_1, \qquad y = \frac{1}{\sqrt{2}}(y_1 + z_1), \qquad z = \frac{1}{\sqrt{2}}(y_1 - z_1)$$

we get

$$0 = x^2 + yz = x_1^2 + \frac{1}{2}\,y_1^2 - \frac{1}{2}\,z_1^2,$$

which the equation of a conical surface.

Example 5.10 *Find the type and position of the conical surface, which is given by the equation*

$$4x^2 - y^2 + 9z^2 + 16x + 6y + 18z + 16 = 0.$$

By some simple manipulations,

$$\begin{aligned} 0 &= 4x^2 + 16x - y^2 + 6y + 9z^2 + 18z + 16 \\ &= 4\{x^2 + 4x + 4 - 4\} - \{y^2 - 6y + 9 - 9\} + 9\{z^2 + 2z + 1 - 1\} + 16 \\ &= 4(x+2)^2 - 16 - (y-3)^2 + 9 + 9(z+1)^2 - 9 + 16 \\ &= 4(x+2)^2 - (y-3)^2 + 9(z+1)^2, \end{aligned}$$

which is the equation of a conical surface.

Example 5.11 *Find the type and position of the conical surface, which is given by the equation*

$$4x^2 + 4y^2 + z^2 - 8xy + 4xz - 4yz - 36x + 18y + 90 = 0.$$

The matrix corresponding to the terms of second order is

$$\mathbf{A} = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix},$$

of the characteristic polynomial

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 4-\lambda & -4 & 2 \\ -4 & 4-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & -4 & 2 \\ 0 & -\lambda & -2\lambda \\ 2 & -2 & 1-\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} 4-\lambda & -4 & 2 \\ 0 & 1 & 2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 4-\lambda & 0 & 10 \\ 0 & 1 & 2 \\ 2 & 0 & 5-\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} 4-\lambda & 10 \\ 2 & 5-\lambda \end{vmatrix} = -\lambda\{\lambda^2 - 9\lambda\} = -\lambda^2(\lambda - 9). \end{aligned}$$

If $\lambda_1 = 0$, then

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence $\mathbf{v}_1 = (1,1,0)$ and $\mathbf{v}_2 = (0,1,2)$ are two linearly independent eigenvectors, where $\|\mathbf{v}_1\| = \sqrt{2}$, and we have (by Gram-Schmidt)

$$\mathbf{v}_2 - \frac{1}{\|\mathbf{v}_1\|^2}(\mathbf{v}_2 \cdot \mathbf{v}_1)\mathbf{v}_1 = (0,1,2) - \frac{1}{2} \cdot 1 \cdot (1,1,0) = \frac{1}{2}(-1,1,4).$$

It follows from $\|(-1,1,4)\| = \sqrt{1+1+16} = 3\sqrt{2}$ that the orthonormed eigenvectors are

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}}(1,1,0) \qquad \text{and} \qquad \mathbf{q}_2 = \frac{1}{3\sqrt{2}}(-1,1,4).$$

If $\lambda_2 = 9$, then

$$\begin{aligned}\mathbf{A} - \lambda_2 \mathbf{I} &= \begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -4 \\ 0 & -9 & -18 \\ 0 & -9 & -18 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

An eigenvector is e.g. $\mathbf{v}_3 = (2, -2, 1)$ of length 3, hence

$$\mathbf{q}_3 = \frac{1}{3}(2, -2, 1).$$

Applying the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

we get

$$\begin{aligned}-36x + 18y &= \left(-\frac{36}{\sqrt{2}} + \frac{18}{\sqrt{2}}\right) x_1 + \left(\frac{36}{3\sqrt{2}} + \frac{18}{3\sqrt{2}}\right) y_1 + \left(-\frac{72}{3} - \frac{36}{3}\right) z_1 \\ &= -9\sqrt{2}\, x_1 + 9\sqrt{2}\, y_1 - 36\, z_1,\end{aligned}$$

and the equation is transferred into

$$\begin{aligned}0 &= 0 \cdot x_1^2 + 0 \cdot y_1^2 + 9z_1^2 - 9\sqrt{2}(x_1 - y_1) - 36z_1 + 90 \\ &= 9\{z_1^2 - 4z_1 + 4 - 4\} - 9\sqrt{2}(x_1 - y_1) + 90,\end{aligned}$$

which we reduce to

$$(z_1 - 2)^2 - \sqrt{2}(x_1 - y_1) + 6 = 0,$$

hence to

$$\frac{1}{\sqrt{2}}(x_1 - y_1) - 3 = \frac{1}{2}(z_1 - 2)^2.$$

We see that we shall perform another change of variables (a rotation) of the eigenspace corresponding to $\lambda_1 = 0$. If we, however, put

$$x_2 = \frac{1}{\sqrt{2}}(x_1 - y_1),$$

then we obviously beg a parabolic cylinder surface.

Example 5.12 *Find the equations of the conical surfaces which are obtained when the conical section*

$$11x^2+4y^2-24xy-20x+40y-60=0, \qquad z=0,$$

is rotated either around the first axis or the second axis.

The corresponding matrix in the XY-plane is given by

$$\mathbf{A}=\begin{pmatrix} 11 & -12 \\ -12 & 4 \end{pmatrix}$$

of the characteristic polynomial

$$\det(\mathbf{A}-\lambda\mathbf{I})=(\lambda-11)(\lambda-4)-144=\lambda^2-15\lambda-100=(\lambda+5)(\lambda-20).$$

The eigenvalues are $\lambda_1=-5$ and $\lambda_2=20$.

If $\lambda_1=-5$, then

$$\mathbf{A}-\lambda_1\mathbf{I}=\begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix}\sim\begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix}.$$

A corresponding normed eigenvector is $\mathbf{q}_1=\frac{1}{5}(3,4)$.

If $\lambda_2=20$, then

$$\mathbf{A}-\lambda_2\mathbf{I}=\begin{pmatrix} -9 & -12 \\ -12 & -16 \end{pmatrix}\sim\begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}.$$

A corresponding normed eigenvector is $\mathbf{q}_2=\frac{1}{5}(-4,3)$.

The transformation matrix is

$$\mathbf{Q}=\frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} x \\ y \end{pmatrix}=\mathbf{Q}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\frac{1}{5}\begin{pmatrix} 3x_1-4y_1 \\ 4x_1+3y_1 \end{pmatrix},$$

thus

$$-20x+40y=-60x_1+80y_1+160x_1+120y_1=100x_1+200y_1.$$

The equation is by the transformation transferred into

$$\begin{aligned} 0 &= -5x_1^2+20y_1^2+100x_1+200y_1-60 \\ &= -5(x_1^2-20x_1+100-100)+20(y_1^2+10y_1+25-25)-60 \\ &= -5(x_1-10)^2+20(y_1+5)^2+500-500-60, \end{aligned}$$

which we reduce to

$$-\frac{1}{12}(x_1-10)^2+\frac{1}{3}(y_1+5)^2=1.$$

This equation describes an hyperbola in the X_1Y_1-plane of centrum $(x_1,y_1)=(10,-5)$.

Remark 5.1 Unfortunately the word "axis" is ambiguous. Here we mean the X_1-axis and the Y_1-axis, i.e. the axes of the hyperbola and not the axes of the original coordinate system. $\diamondsuit$

When the hyperbola is rotated around the X_1-axis, we obtain an hyperboloid of 1 sheet.

When the hyperbola is rotated around the Y_1-axis, we obtain an hyperboloid of 2 sheets.

Example 5.13 *A conical surface is in a rectangular XYZ-coordinate system given by the equation*

$$5x^2+5y^2+2xy-2z^2+4\sqrt{2}(x-y)+4z-2=0.$$

Indicate the type of the conical surface, the centrum of the conical surface and the directions of the axes in the XYZ-coordinate system.

The corresponding matrix

$$\mathbf{A}=\begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

has the characteristic e polynomial

$$\det(\mathbf{A}-\lambda\mathbf{I})=-(\lambda+2)\{(\lambda-5)^2-1\}=-(\lambda+2)(\lambda-4)(\lambda-6),$$

of the roots $\lambda_1=6$, $\lambda_2=4$, $\lambda_3=-2$.

If $\lambda_1=6$, we get the eigenvector $\mathbf{q}_1=\dfrac{1}{\sqrt{2}}(1,1,0)$.
If $\lambda_2=4$, we get the eigenvector $\mathbf{q}_2=\dfrac{1}{\sqrt{2}}(-1,1,0)$.
If $\lambda_3=-2$, we get the eigenvector $\mathbf{q}_3=(0,0,1)$,
all three normed, hence the transformation matrix is

$$\mathbf{Q}=\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{2}}(x_1-y_1) \\ \frac{1}{\sqrt{2}}(x_1+y_1) \\ z_1 \end{pmatrix},$$

and

$$4\sqrt{2}\,(x-y)=4\sqrt{2}\cdot\frac{1}{\sqrt{2}}\cdot(-2y_1)=-8y_1.$$

The equation is by the transformation transferred into

$$\begin{aligned} 0 &= 6x_1^2+4y_1^2-2z_1^2-8y_1+4z-2 \\ &= 6x_1^2+4(y_1^2-2y_1+1-1)-2(z_1^12-2z+1-1)-2 \\ &= 6x_1^2+4(y_1-1)^2-2(z_1-1)^2-4+2-2, \end{aligned}$$

hence by a rearrangement,

$$\frac{3}{2}\,x_1^2+(y_1-1)^2-\frac{1}{2}\,(z_1-1)^2=1.$$

It follows from the structure that the conical surface hyperboloid of 1 sheet, and centrum $(x_1,y_1,z_1)=(0,1,1)$, thus in XYZ-coordinates,

$$(x,y,z)=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1\right).$$

6 Quadratic forms

Example 6.1 . *Given the quadratic form*

(3) $\quad (a+1)x^2 + (a+1)y^2 + 2axy + 2az^2,$

where a is a real number.

1. *Find the matrix of the form matrix $\mathbf{A}$, and the eigenvalues of $\mathbf{A}$.*
2. *Find the values of a, for which $\mathbf{A}$ has three mutually different eigenvalues.*

   Reduce (3) to a quadratic form without product terms for these values of a, and find a reducing proper orthogonal substitution.
3. *Describe for $a = 0$ and $a = -\frac{1}{4}$ the type of the conical surface which is an ordinary rectangular coordinate system XYZ of positive orientation is described by the equation*

   $$(a+1)x^2 + (a+1)y^2 + 2axy + 2az^2 + \sqrt{2}(x-y) + 4z = 0.$$

   In case of $a = -\frac{1}{4}$, find a parametric description in the XYZ-system for the symmetry axis through the vertices of the surface.

1. The matrix is

   $$\mathbf{A} = \begin{pmatrix} a+1 & a & 0 \\ a & a+1 & 0 \\ 0 & 0 & 2a \end{pmatrix}.$$

   The characteristic polynomial is

   $$\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - 2a)\{(\lambda - a - 1)^2 - a^2\} = -(\lambda - 1)(\lambda - 2a - 1)(\lambda - 2a).$$

   The eigenvalues are

   $$\lambda_1 = 1, \qquad \lambda_2 = 2a+1, \lambda_3 = 2a.$$

2. It follows that $\lambda_2 \neq \lambda_3$ for every a. Furthermore, $\lambda_1 \neq \lambda_2$ for $a \neq 0$, and $\lambda_1 \neq \lambda_3$ for $a \neq \dfrac{1}{2}$. Hence, we have three mutually different eigenvalues, when

   $$a \notin \left\{0, \frac{1}{2}\right\}.$$

   Assume that $a \notin \{0, \frac{1}{2}\}$.

   If $\lambda_1 = 1$, then

   $$\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ 0 & 0 & 2a-1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

   A normed eigenvector is $\mathbf{q}_1 = \dfrac{1}{\sqrt{2}}(1, -1, 0)$.

If $\lambda_2 = 2a+1$, then

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} -a & a & 0 \\ a & -a & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A normed eigenvector is $\mathbf{q}_2 = \dfrac{1}{\sqrt{2}}(1,1,0)$.

If $\lambda_3 = 2a$, then $\mathbf{q}_3 = (0,0,1)$ is trivially a normed eigenvector.

The transformation is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{Q} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \qquad \text{hvor} \qquad \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and then (3) is written

$$x_1^2 + (2a+1)y_1^2 + 2az_1^2, \qquad a \notin \left\{0, \frac{1}{2}\right\}.$$

3. From

$$\sqrt{2}(x-y) + 4z = \sqrt{2} \cdot \sqrt{2}\, x_1 + 4z_1 = 2x_1 + 4z_1,$$

follows by the transformation that the equation becomes

$$\begin{aligned} 0 &= x_1^2 + (2a+1)y_1^2 + 2az_1^2 + 2x_1 + 4z_1 \\ &= \{x_1^2 + 2x_1 + 1 - 1\} + (2a+1)y_1^2 + 2az_1^2 + 4z_1 \\ &= (x_1+1)^2 + (2a+1)y_1^2 + 2az_1^2 + 4z_1 - 1. \end{aligned}$$

(a) If $a = 0$ then by 2) we have an exceptional case. Hoswever, if $a = 0$, then it follows immediately that

$$0 = x^2 + y^2 + \sqrt{2}(x-y) + 4z = \left(x + \frac{1}{\sqrt{2}}\right)^2 + \left(y - \frac{1}{\sqrt{2}}\right)^2 + 4z - 1,$$

which is the equation of an elliptic paraboloid.

(b) If instead $a = -\dfrac{1}{4}$, then

$$\begin{aligned} 0 &= (x_1+1)^2 + \left(-\frac{1}{2} + 1\right) y_1^2 - \frac{1}{2}\, z_1^2 + 4z_1 - 1 \\ &= (x_1+1)^2 + \frac{1}{2}\, y_1^2 - \frac{1}{2}(z_1^2 - 8z_1 + 16) - 1 + 8, \end{aligned}$$

hence

$$-\frac{1}{7}(x_1+1)^2 - \frac{1}{14}\, y_1^2 + \frac{1}{14}(z_1 - 4)^2 = 1,$$

which is the equation of an hyperboloid of 2 sheets.

The vertices of the surface are given by $(z_1-4)^2 = 14$, and $x_1 = -1$, $y_1 = 0$, i.e. a line parallel to the Z_1-axis = the Z-axis.

Since $x_1 = -1$ and $y_1 = 0$ correspond to $x = \dfrac{1}{\sqrt{2}}$ and $y = \dfrac{1}{\sqrt{2}}$, the parametric description of the symmetry axis,

$$(x,y,z) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, t\right), \qquad t \in \mathbb{R}.$$

Example 6.2 *Given the matrix*

$$\mathbf{Q} = \frac{1}{7}\begin{pmatrix} 2 & 3 & 6 \\ 3 & -6 & 2 \\ 6 & 2 & -3 \end{pmatrix}$$

and the equation of second order

(4) $\quad 4x^2 + 9y^2 + 36z^2 + 12xy + 24xz + 36yz + 6x + 2y - 3z = 0.$

1. *Prove that* $\mathbf{Q}$ *is proper and orthogonal and that one can reduce the quadratic form in (6.2) to a form without product terms by applying* $\mathbf{Q}$ *as the matrix of change of variables.*
2. *Prove that (4) in an ordinary rectangular coordinate system* XYZ *in space of positive orientation describes a cylinder surface, and describe the type and the direction of the generator of this surface.*

1. It follows from $2^2+3^2+6^2 = 4+9+36 = 49 = 7^2$, that all columns in $\mathbf{Q}$ are unit vectors. Then we calculate the three possible inner products [without the factor $\frac{1}{7}$], to get

$$\begin{aligned} (2,3,6)\cdot(3,-6,2) &= 6-18+12 = 0, \\ (2,3,6)\cdot(6,2,-3) &= 12+6-18 = 0, \\ (3,-6,2)\cdot(6,2,-3) &= 18-12-6 = 0. \end{aligned}$$

Hence, it follows that $\mathbf{Q}$ is proper orthogonal.

To (4) corresponds the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 6 & 12 \\ 6 & 9 & 18 \\ 12 & 18 & 36 \end{pmatrix}.$$

It follows from

$$\mathbf{A}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 12 \\ 6 & 9 & 18 \\ 12 & 18 & 36 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 98 \\ 147 \\ 294 \end{pmatrix} = 49\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix},$$

$$\mathbf{A}\begin{pmatrix}3\\-6\\2\end{pmatrix}=\begin{pmatrix}4&6&12\\6&9&18\\12&18&36\end{pmatrix}\begin{pmatrix}3\\-6\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}=0\begin{pmatrix}3\\-6\\2\end{pmatrix},$$

$$\mathbf{A}\begin{pmatrix}6\\2\\-3\end{pmatrix}=\begin{pmatrix}4&6&12\\6&9&18\\12&18&36\end{pmatrix}\begin{pmatrix}6\\2\\-3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}=0\begin{pmatrix}6\\2\\-3\end{pmatrix},$$

that the first column in $\mathbf{Q}$ is an eigenvector for $\mathbf{A}$ corresponding to the eigenvalue $\lambda_1 = 49$, and the latter two columns are both (orthogonal) eigenvectors corresponding to the eigenvalue $\lambda_2 = 0$.

It follows from

$$\begin{aligned}6x+2y-3z &= \frac{1}{7}\{6(2x_1+3y_1+6z_1)+2(3x_1-6y_1+2z_1)-3(6x_1+2y_1-3z_1)\}\\ &= \frac{1}{7}\{(12+6-18)x_1+(18-12-6)y_1+(36+4+9)z_1\}=7z_1,\end{aligned}$$

that (6.2) by the transformation is transferred into

$$0=49x_1^2+7z_1, \qquad \text{thus} \qquad z_1=-7x_1^2 \text{ or } x_1^2=-\frac{1}{7}\,z_1.$$

2. It follows from the result of 1) that (6.2) describes a parabolic cylinder surface with the Y_1-axis as the direction of the generators. In the XYZ-space the Y_1-axis is given by the eigenvector $(3,-6,2)$, so this indicates the direction of the generators.

Example 6.3 *Find the type of the conical surface which is described by the equation*

$$3x^2 - 3y^2 + 12xz + 12yz + 4x - 4y - 2z = 0,$$

where (x, y, z) are coordinates in an ordinary rectangular coordinate system $(O; \vec{i}, \vec{j}, \vec{k})$ in space of positive orientation. Find the equations of the symmetry planes of the conical surface.

The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 6 \\ 0 & -3 & 6 \\ 6 & 6 & 0 \end{pmatrix}$$

has the characteristic polynomial

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 3-\lambda & 0 & 6 \\ 0 & -3-\lambda & 6 \\ 6 & 6 & -\lambda \end{vmatrix} = -(\lambda-3)\begin{vmatrix} \lambda+3 & -6 \\ -6 & \lambda \end{vmatrix} + 6\begin{vmatrix} 0 & -(\lambda+3) \\ 6 & 6 \end{vmatrix} \\ &= -(\lambda-3)(\lambda^2+3\lambda-36) + 36(\lambda+3) \\ &= -\lambda^3 - 3\lambda^2 + 36\lambda + 3\lambda^2 + 9\lambda - 108 + 36\lambda + 108 \\ &= -\lambda^3 + 81\lambda = -\lambda(\lambda-9)(\lambda+9). \end{aligned}$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 9$ and $\lambda_3 = -9$.

If $\lambda_1 = 0$, then

$$\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} 3 & 0 & 6 \\ 0 & -3 & 6 \\ 6 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

An eigenvector is $(2, -2, -1)$ of length 3, hence a normed eigenvector is $\mathbf{q}_1 = \frac{1}{3}(2, -2, -1)$.

If $\lambda - 2 = 9$, then

$$\mathbf{A} - \lambda_2\mathbf{I} = \begin{pmatrix} -6 & 0 & 6 \\ 0 & -12 & 6 \\ 6 & 6 & -9 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 2 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is $(2, 1, 2)$ of length 3, hence a normed eigenvector is $\mathbf{q}_2 = \frac{1}{3}(2, 1, 2)$.

If $\lambda_3 = -9$, then

$$\mathbf{A} - \lambda_3\mathbf{I} = \begin{pmatrix} 12 & 0 & 6 \\ 0 & 6 & 6 \\ 6 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is $(1, 2, -2)$ of length 3, hence a normed eigenvector is $\mathbf{q}_3 = \frac{1}{3}(1, 2, -2)$.

The transformation is fixed by the matrix

$$\mathbf{Q} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & -2 \end{pmatrix}.$$

If follows from

$$\begin{aligned}4x-4y-2z &= \frac{1}{3}\{4(2x_1+2y_1+z_1)-4(-2x_1+y_1+2z_1)-2(-x_1+2y_1-2z_1)\}\\ &= \frac{1}{3}\{(8+8+2)x_1+(8-4-4)y_1+(4-8+4)z_1\}=6x_1,\end{aligned}$$

that by this transformation the equation is transferred into

$$9y_1^2-9z_1^2+6x_1=0, \qquad \text{thus} \qquad \frac{2}{3}x_1=z_1^2-y_1^2,$$

which is the equation of an hyperbolic paraboloid with the X_1Z_1-plane and X_1Y_1-plane as planes of symmetry. The normal of the X_1Z_1-plane is the eigenvector $(2,1,2)$, which indicates the direction of the Y_1-axis, i.e. an equation is

$$2x+y+2z=0.$$

The normal of the X_1Y_1-plane is the direction of the Z_1-axis, thus $(1,2,-2)$, hence an equation of this plane is

$$x+2y-2z=0.$$

Example 6.4 *1. Reduce the quadratic form*

$$5x^2+5y^2+5z^2-8xy-8yz-8zx$$

to a form $\lambda_1x_1^2+\lambda_2y_1^2+\lambda_3z_1^2$, where $\lambda_1\le\lambda_2\le\lambda_3$, and indicate a proper orthogonal substitution, which performs the reduction.

2. *A conical surface is in the coordinates x, y, z with respect to an ordinary rectangular coordinate system $(O;\vec{i},\vec{j},\vec{k})$ in space of positive orientation given by the equation*

$$5x^2+5y^2+5z^2-8xy-8yz-8zx-6x-6y-6z=9.$$

Apply 1) to prove that the surface is a rotational cone. Find in the coordinates with respect to the system $(O;\vec{i},\vec{j},\vec{k})$ the vertex of the cone and a parametric description of the axis of rotation.

3. *Prove that the point $P:(1,1-\sqrt{2},1+\sqrt{2})$ lies on the conical surface. Find an equation of the plane, which contains both the axis of rotation and the generator through P.*

1. The matrix

$$\mathbf{A}=\begin{pmatrix}5&-4&-4\\-4&5&-4\\-4&-4&5\end{pmatrix}$$

has the characteristic polynomial

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & -4 & -4 \\ -4 & 5-\lambda & -4 \\ -4 & -4 & 5-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda-3 & -\lambda-3 & -\lambda-3 \\ 0 & 9-\lambda & -9+\lambda \\ -4 & -4 & 5-\lambda \end{vmatrix}$$

$$= (\lambda+3)(\lambda-9)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -4 & -4 & 5-\lambda \end{vmatrix} = (\lambda+3)(\lambda-9)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 9-\lambda \end{vmatrix}$$

$$= -(\lambda+3)(\lambda-9)^2,$$

hence the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = \lambda_3 = 9$.

If $\lambda_1 = -3$, then

$$\mathbf{A}-\lambda_1\mathbf{I} = \begin{pmatrix} 8 & -4 & -4 \\ -4 & 8 & -4 \\ -4 & -4 & 8 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -1 \\ -2 & 4 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is $\mathbf{v}_1 = (1,1,1)$ of length $\sqrt{3}$, hence a normed eigenvector is

$$\mathbf{q}_1 = \frac{1}{\sqrt{3}}(1,1,1).$$

If $\lambda_2 = \lambda_3 = 9$, then

$$\mathbf{A}-\lambda_2\mathbf{I} = \begin{pmatrix} -4 & -4 & -4 \\ -4 & -4 & -4 \\ -4 & -4 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Two linearly independent eigenvectors are $\mathbf{v}_2 = (1,0,-1)$ and $\mathbf{v}_3 = (0,1,-1)$. We get by the Gram-Schmidt method

$$\mathbf{v}_3 - \frac{1}{\|\mathbf{v}_2\|^2}(\mathbf{v}_3\cdot\mathbf{v}_2)\mathbf{v}_2 = (0,1,-1) - \frac{1}{2}(1,0,-1) = \frac{1}{2}(-1,2,-1),$$

hence two orthonormed eigenvectors are

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}(1,0,-1) \quad \text{and} \quad \mathbf{q}_3 = \frac{1}{\sqrt{6}}(1,-2,1).$$

We can choose the transformation matrix as

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

and the form becomes

$$-3x_1^2 + 9y_1^2 + 9z_1^2.$$

2. Now,

$$\begin{aligned}
&-6x-6y-6z\\
&\quad=-6\left\{\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)x_1+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)y_1+\left(\frac{1}{\sqrt{6}}-\frac{2}{\sqrt{6}}+\frac{1}{\sqrt{6}}\right)z_1\right\}\\
&\quad=-6\sqrt{3}\,x_1,
\end{aligned}$$

so the equation is transferred by the transformation into

$$\begin{aligned}
9 &= -3x_1^2+9y_1^2+9z_1^2-6\sqrt{3}\,x_1\\
&= -3\left\{x_1^2+2\sqrt{3}\,x_1+3-3\right\}+9y_1^2+9z_1^2\\
&= -3(x_1+\sqrt{3})^2+9y_1^2+9z_1^2+9,
\end{aligned}$$

which is reduced to

$$-(x+\sqrt{3})^2+3y_1^2+3z_1^2=0 \quad \text{or} \quad (x_1+\sqrt{3})^2=3(y_1^2+z_1^2).$$

This equation describes a conical rotational cone of vertex at $(x_1,y_1,z_1)=(-\sqrt{3},0,0)$, which in the XYZ-space is given by the coordinates

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\mathbf{Q}\begin{pmatrix}-\sqrt{3}\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}-1\\ -1\\ -1\end{pmatrix}.$$

The axis of rotation is the X_1-axis. Its direction in the XYZ-space given by a constant times the first column of $\mathbf{Q}$, thus a parametric description is

$$t(1,1,1), \qquad t \in \mathbb{R}.$$

3. It follows from

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \mathbf{Q}^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 1+\sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ -1 \\ \sqrt{3} \end{pmatrix}$$

and

$$(x_1+\sqrt{3})^2 = 4 \cdot 3 = 12, \qquad 3(y_1^2+z_1^2) = 3(1+3) = 12,$$

that $P; (1, 1-\sqrt{2}, 1+\sqrt{2})$ lies on the surface.

The generator in the XYZ-space is given by

$$(1, 1-\sqrt{2}, 1+\sqrt{2}) - (-1,-1,-1) = (2, 2-\sqrt{2}, 2+\sqrt{2}) = \sqrt{2}(\sqrt{2}, \sqrt{2}-1, \sqrt{2}+1).$$

A normal is given by

$$(1,1,1) \times (\sqrt{2}, \sqrt{2}-1, \sqrt{2}+1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ \sqrt{2}, \sqrt{2}-1, \sqrt{2}+1 & & \end{vmatrix}$$
$$= (\sqrt{2}+1-\sqrt{2}+1, \sqrt{2}-\sqrt{2}-1, \sqrt{2}-1-\sqrt{2}) = (2,-1,-1),$$

thus the equation of the plane through $(-1,-1,-1)$ is

$$0 = (2,-1,-1) \cdot (x+1, y+1, z+1) = 2x+2-y-1-z-1 = 2x-y-z,$$

hence

$$2x - y - z = 0.$$

Example 6.5 *Given a conical surface in ordinary rectilinear coordinates in space by the equation*

$$ax^2 + y^2 + z^2 + 6yz = 1, \qquad \textit{where } a \in \mathbb{R}.$$

1. *Find a quadratic form* $\lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 z_1^2$, *which can be reduced to the quadratic form occurring on the left hand side of the equation by some orthogonal substitution.*

   (The orthogonal substitution is not requested.)

2. *Find all* a, *for which the conical surface is a rotational surface. Find for each such value of* a *the type of the surface and also a parametric description of the rotational axis in the given coordinates.*

1. The corresponding matrix

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$$

has the characteristic polynomial

$$\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - a)\{(\lambda - 1)^2 - 3^2\} = -(\lambda - a)(\lambda + 2)(\lambda - 4).$$

The eigenvalues are $\lambda_1 = a$, $\lambda_2 = -2$ and $\lambda_3 = 4$.

It follows from the above that one can reduce to

$$ax_1^2 - 2y_1^2 + 4z_1^2 = 1.$$

Remark 6.1 It is not difficult to show that the orthogonal substitution, which we shall use, can be chosen as

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad \Diamond$$

2. If $ax_1^2 - 2y_1^2 + 4z_1^2 = 1$ describes a rotational surface, then either $a = -2$ or $a = 4$.

   (a) If $a = -2$, then

   $$-2(x_1^2 + y_1^2) + 4z_1^2 = 1,$$

   which is the equation of a rotational hyperboloid of 2 sheets. The axis of rotation is the Z_1-axis.

   (b) If $a = 4$, then

   $$4(x_1^2 + z_1^2) - 2y_1^2 = 1,$$

   which is the equation of a rotational hyperboloid of 1 sheet. The axis of rotation is the Y_1-axis.

Example 6.6 *Given a conical surface in ordinary rectangular coordinates in space by the equation*

$$4xy + az^2 = 1, \qquad \textit{where} \qquad a \in \mathbb{R}.$$

1. *Find a quadratic form* $\lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 z_1^2$*, which the quadratic form, occurring on the left hand side of the equation can be reduced to by an application of some orthogonal substitution.*

   (The orthogonal substitution is not requested).

2. *Find all* a*, for which the given conical surface is a rotational surface and indicate for these values the type of the surface and a parametric description of its axis of rotation in the given coordinates* x*,* y*,* z*.*

3. *Prove that there is precisely one value of* a*, for which the surface is a cylindric surface, and describe its type and its axis of symmetry.*

1. The corresponding matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$$

has the characteristic polynomial

$$(\lambda - 2)(\lambda + 2)(\lambda - a),$$

thus the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -2$ and $\lambda_3 = a$. Hence, one can in some other coordinates reduce to

$$2x_1^2 - 2y_1^2 + az_1^2 = 1.$$

Remark 6.2 It is quite easy to prove that one may choose' the orthogonal substitution as

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \Diamond$$

2. The form describes a rotational surface, if $a = \pm 2$.

   (a) If $a = 2$, then

   $$2(x_1^2 + z_1^2) - 2y_1^2 = 1,$$

   which is the equation of a rotational hyperboloid of 1 sheet. The axis of rotation is the Y_1-axis, i.e. in XYZ-space

   $$(-t, t, 0), \qquad t \in \mathbb{R}.$$

   (b) If $a = -2$, then

   $$2x_1^2 - 2(y_1^2 + z_1^2) = 1,$$

   which is the equation of a rotational hyperboloid of 2 sheets and with the X_1-axis as its axis of rotation. In XYZ-space the X_1-axis is given by

   $$(t, t, 0), \qquad t \in \mathbb{R}.$$

3. It follows that we get a cylindric surface for $a = 0$,

$$2x_1^2 - 2y_1^2 = 1.$$

The direction of the generator is the Z_1-axis, i.e. $(0, 0, t)$, $t \in \mathbb{R}$, in the original coordinates.

Example 6.7 *Given in an ordinary rectangular coordinate system in space of positive orientation a conical surface by the equation*

$$x^2 - 4x - 2z^2 - 4z - 6y = 0.$$

Find the type and position of the surface.
Find all generators through $(0,0,0)$.

We get by using some suitable reformulations,

$$\begin{aligned} 0 &= (x^2 - 4x + 4 - 4) - 2(z^2 + 2z + 1 - 1) - 6y \\ &= (x-2)^2 - 4 - 2(z+1)^2 + 2 - 6y, \end{aligned}$$

hence

$$6\left(y + \frac{1}{3}\right) = (x-2)^2 - 2(z+1)^2.$$

This equation describes an hyperbolic paraboloid.

Then we transform into the canonical form

$$z_1 = y + \frac{1}{3} = \frac{x_1^2}{(\sqrt{6})^2} - \frac{y_1^2}{(\sqrt{3})^2} = \frac{(x-2)^2}{6} - \frac{(z+1)^2}{3},$$

thus

$$x_1 = x - 2, \quad y_1 = z + 1, \quad z_1 = y + \frac{1}{3}, \quad a = \sqrt{6}, \quad b = \sqrt{3}.$$

It follows that we obtain the systems of straight lines on the surface

$$\left\{ \begin{array}{l} \dfrac{x_1}{a} + \dfrac{y_1}{b} = k, \\ \dfrac{k}{a}\, x_1 - \dfrac{k}{b}\, y_1 - z_1 = 0, \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{l} \dfrac{x_1}{a} - \dfrac{y_1}{b} = k, \\ \dfrac{k}{a}\, x_1 + \dfrac{k}{b}\, y_1 - z_1 = 0. \end{array} \right.$$

The first of these is

$$\frac{x-2}{\sqrt{6}} + \frac{z+1}{\sqrt{3}} = k \quad \text{and} \quad \frac{k}{\sqrt{6}}(x-2) - \frac{k}{\sqrt{3}}(z+1) - \left(y + \frac{1}{3}\right) = 0.$$

If $(x,y,z) = (0,0,0)$, then $k = -\dfrac{2}{\sqrt{6}} + \dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{2}-1}{\sqrt{3}}$, thus

$$\begin{aligned} 0 &= -\frac{\sqrt{2}-1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}}(x-2) + \frac{\sqrt{2}-1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}(z+1) - \left(y + \frac{1}{3}\right) \\ &= -\frac{\sqrt{2}-1}{3\sqrt{2}}(x-2) + \frac{\sqrt{2}-1}{3}(z+1) - \left(y + \frac{1}{3}\right), \end{aligned}$$

so multiplying by $-3\sqrt{2}$,

$$\begin{aligned} 0 &= (\sqrt{2}-1)(x-2) - (2-\sqrt{2})(z+1) + 3\sqrt{2}\left(y + \frac{1}{3}\right) \\ &= (\sqrt{2}-1)x - (2-\sqrt{2})z + 3\sqrt{2}\,y - 2\sqrt{2} + 2 - 2 + \sqrt{2} + \sqrt{2} \\ &= (\sqrt{2}-1)x - (2-\sqrt{2})z + 3\sqrt{2}y. \end{aligned}$$

If we multiply by $\sqrt{2}+1$, we get instead

$$0 = x - \sqrt{2}\,z + 3(2+\sqrt{2})y,$$

which is slightly nicer.

The second family is then written

$$\frac{x-2}{\sqrt{6}} - \frac{z+1}{\sqrt{3}} = k \quad \text{and} \quad \frac{k}{\sqrt{6}}(x-2) + \frac{k}{\sqrt{3}}(z+1) - \left(y+\frac{1}{3}\right) = 0.$$

If $(x,y,z) = (0,0,0)$, then $k = -\dfrac{2}{\sqrt{6}} - \dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{2}+1}{\sqrt{3}}$, hence

$$0 = -\frac{\sqrt{2}+1}{3\sqrt{2}}(x-2) - \frac{\sqrt{2}+1}{3}(z+1) - \left(y+\frac{1}{3}\right),$$

so multiplying by $-3\sqrt{2}$,

$$\begin{aligned} 0 &= (\sqrt{2}+1)(x-2) + (2+\sqrt{2})(z+1) + 3\sqrt{2}\left(y+\frac{1}{3}\right) \\ &= (\sqrt{2}+1)x + (2+\sqrt{2})z + 3\sqrt{2}\,y - 2\sqrt{2} - 2 + 2 + \sqrt{2} + \sqrt{2} \\ &= (\sqrt{2}+1)x + (2+\sqrt{2})z + 3\sqrt{2}\,y. \end{aligned}$$

Then if we multiply by $\sqrt{2}-1$, we obtain the equivalent

$$0 = x + \sqrt{2}\,z + 3(2-\sqrt{2})y.$$

The two generators through $(0,0,0)$ are

$$0 = x - \sqrt{2}\,x + 3(2+\sqrt{2})y \quad \text{and} \quad 0 = x + \sqrt{2}\,z + 3(2-\sqrt{2})y.$$

Example 6.8 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

1. *Prove that $\lambda = 1$ is an eigenvalue of $\mathbf{A}$, and find all eigenvectors corresponding to this eigenvalue.*
2. *Find every eigenvalue of $\mathbf{A}$.*
3. *Consider in an ordinary rectangular coordinate system in space of positive orientation the quadratic equation*

$$2x^2 + 5y^2 + 2z^2 + 4xy + 2xz + 4yz = 1.$$

Prove that this equation describes a rotational ellipsoid, and indicate an equation of that plane, which is perpendicular to the axis of rotation through the centrum of the ellipsoid.

1. It follows from

$$\mathbf{A} - 1 \cdot \mathbf{I} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

that $\lambda_1 = 1$ is an eigenvalue of rank 2. Two linearly independent eigenvectors are e.g. $\mathbf{v}_1 = (1,0,-1)$ and $\mathbf{v}_2 = (1,-1,1)$ where $\|\mathbf{v}_1 = \sqrt{2}$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, and $\|\mathbf{v}_2\| = \sqrt{3}$, thus the subspace of all eigenvectors corresponding to $\lambda_1 = 1$ is spanned by the orthonormal eigenvectors

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}}(1,0,-1) \qquad \text{and} \qquad \mathbf{q}_2 = \frac{1}{\sqrt{3}}(1,-1,1).$$

2. It follows from the construction above that $(1,2,1)$ must be perpendicular to both $\mathbf{q}_1$ and $\mathbf{q}_2$- Now, since

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

we see that $\mathbf{q}_3 = \dfrac{1}{\sqrt{6}}(1,2,1)$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = 7$.

Remark 6.3 Since tr $\mathbf{A} = 2+5+2 = 9 = \lambda_1+\lambda_2+\lambda_3 = 2+\lambda_3$, we derive once more that $\lambda_3 = 9-2 = 7.\Diamond$

Hence, all eigenvalues are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 7$.

3. If we apply the transformation matrix

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix},$$

we transfer the quadratic equation into

$$x_1^2 + y_1^2 + 7z_1^2 = 1,$$

which describes a rotational ellipsoid. The plane which is perpendicular to the rotational axis, i.e. the Z_1-axis, must be perpendicular to $(1,2,1)$ and pass through $(0,0,0)$, hence its equation is $x+2y+z=0$.

Example 6.9 *Given in an ordinary rectangular coordinate system XYZ in space of positive orientation a conical surface by the equation*

$$ax^2+ay^2+az^2+6xy+8xz = 1,$$

where a is any real number.

1. *Find for every a the type of the surface.*
2. *Put $a=1$, and prove that the surface intersects the XY-plane in an hyperbola.*
3. *Find a parametric description of each of the asymptotes of the hyperbola.*

1. The corresponding matrix

$$\mathbf{A} = \begin{pmatrix} a & 3 & 4 \\ 3 & a & 0 \\ 4 & 0 & a \end{pmatrix}$$

has the characteristic polynomial

$$\begin{aligned} \det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} a-\lambda & 3 & 4 \\ 3 & a-\lambda & 0 \\ 4 & 0 & a-\lambda \end{vmatrix} \\ &= 4\begin{vmatrix} 3 & 4 \\ a-\lambda & 0 \end{vmatrix} + (a-\lambda)\begin{vmatrix} a-\lambda & 3 \\ 3 & a-\lambda \end{vmatrix} \\ &= -16(a-\lambda)+(a-\lambda)\{(\lambda-a)^2-9\} \\ &= -(\lambda)\{(\lambda-a)^2-25\} = -(\lambda-a)(\lambda-a-5)(\lambda-a+5). \end{aligned}$$

We have three different eigenvalues,

$$\lambda_1 = a - 5, \qquad \lambda_2 = a, \qquad \lambda_3 = a + 5.$$

In order to find the type of the surface we analyze the signs pf the λ,

$$\lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 z_1^2 = 1 \qquad (> 0).$$

	λ_1	λ_2	λ_3	Art
$a < -5$	$-$	$-$	$-$	The empty set.
$a = -5$	$-$	$-$	0	The empty set.
$-5 < a < 0$	$-$	$-$	$+$	Hyperboloid of 2 sheets.
$a = 0$	$-$	0	$+$	Hyperbolic cylinder.
$0 < a < 5$	$-$	$+$	$+$	Hyperboloid of 1 sheet.
$a = 5$	0	$+$	$+$	Elliptic cylinder.
$a > 5$	$+$	$+$	$+$	Ellipsoid.

2. If $a = 1$, then it follows from the above that we have an hyperboloid of 1 sheet. We get by insertion of $a = 1$ and $z = 0$ that

$$1 = x^2 + y^2 + 6xy = \alpha(x+y)^2 + \beta(x-y)^2,$$

hence $\alpha + \beta = 1$ and $2\alpha - 2\beta = 6$, and whence $\alpha - \beta = 3$. Thus $\alpha = 2$ and $\beta = -1$, so

$$2(x+y)^2 - (x-y)^2 = 1,$$

which clearly is the equation of an hyperbola in the XY-plane with $(0,0)$ as centrum.

3. The asymptotes satisfy

$$\begin{aligned} 0 &= 2(x+y)^2-(x-y)^2=\left(\sqrt{2}\,x+\sqrt{2}\,y\right)^2-(x-y)^2 \\ &= \{(\sqrt{2}+1)x+(\sqrt{2}-1)y\}\{(\sqrt{2}-1)x+(\sqrt{2}+1)y\}, \end{aligned}$$

thus the equations of the asymptotes are

$$(\sqrt{2}+1)x+(\sqrt{2}-1)y=0 \quad \text{or} \quad (\sqrt{2}-1)x+(\sqrt{2}+1)y=0,$$

which can also be written in the form

$$y=-(\sqrt{2}+1)^2x=-(3+2\sqrt{2})x \quad \text{or} \quad y=-(\sqrt{2}-1)^2x=-(3-2\sqrt{2})x.$$

Example 6.10 *Given the matrix*

$$\mathbf{A}=\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & 4 \\ 2 & 4 & 3 \end{pmatrix}.$$

1. *Prove that* $\mathbf{v}=(2\ \ 1\ \ 2)^T$ *is an eigenvector of* $\mathbf{A}$*, and find the corresponding eigenvalue.*
2. *Prove that* $\lambda=-2$ *is an eigenvalue of* $\mathbf{A}$*, and find the eigenvectors corresponding to this eigenvalue .*
3. *Find a proper orthogonal matrix* $\mathbf{Q}$ *and a diagonal matrix* $\mathbf{\Lambda}$*, such that*

$$\mathbf{Q}^T\mathbf{A}\mathbf{Q}=\mathbf{\Lambda}.$$

4. *Find the type of the conical surface which is given by the equation*

$$6x^2+3y^2+3z^2-4xy+4xz+8yz=14,$$

where (x,y,z) *are the coordinates in an ordinary rectangular coordinate system in space of positive orientation.*

1. We get by insertion,

$$\mathbf{A}\mathbf{v}=\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & 4 \\ 2 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}=\begin{pmatrix} 14 \\ 7 \\ 14 \end{pmatrix}=7\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

thus $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue $\lambda_1=7$. Then by norming,

$$\mathbf{q}_1=\frac{1}{3}(2,1,2).$$

2. It follows from

$$\mathbf{A}-\lambda_3\mathbf{I}=\begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix} \sim \begin{pmatrix} 0 & 18 & 18 \\ 0 & 9 & 9 \\ 2 & 4 & 5 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

that $\mathbf{v}_3 = (1, 2, -2)$ of length 3 is eigenvector corresponding to $\lambda_3 = -2$. Then by norming,

$$\mathbf{q}_3 = \frac{1}{3}(1, 2, -2).$$

3. Since

$$(2,1,2)\times(1,2,-2) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix} = (-6,6,3) = -3(2,-2,-1)$$

is perpendicular to both $\mathbf{q}_1$ and $\mathbf{q}_3$, and $\mathbf{Q}$ is symmetric, hence $\mathbf{v}_2 = (2,-2,-1)$ must be an eigenvector,

$$\mathbf{A}\mathbf{v}_3 = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & 4 \\ 2 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 14 \\ -14 \\ -7 \end{pmatrix} = 7\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix},$$

i.e. $\mathbf{q}_2 = \frac{1}{3}(2,-2,-1)$ is a normed eigenvector corresponding to $\lambda_2 = 7$, which is even orthogonal to the eigenvector $\mathbf{q}_1$ for the eigenvalue $\lambda = 7$. Hence

$$\mathbf{Q} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{pmatrix}.$$

4. Applying the transformation given by $\mathbf{Q}$ the equation is transferred into

$$7x_1^2 + 7y_1^2 - 2z_1^2 = 1,$$

which is the equation of an hyperboloid of 1 sheet.

Example 6.11 *Given in an ordinary rectangular coordinate system in space a surface of the equation*

$$2x^2 + y^2 + z^2 + 4yz = 1.$$

Describe the type of this surface.

The corresponding matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

has the characteristic polynomial

$$\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda-2)\{(\lambda-1)^2-2^2\} = -(\lambda+1)(\lambda-3)(\lambda-2)$$

of the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 3$. By an orthogonal transformation the equation is transferred into

$$-x_1^2 + 2y_1^2 + 3z_1^2 = 1,$$

which is the equation of an hyperboloid of 1 sheet.

Example 6.12 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 7 & 2 \\ 4 & 2 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

1. *Prove that* $\mathbf{v}$ *is an eigenvector of* $\mathbf{A}$.
2. *Solve the equation* $\mathbf{v}^T\mathbf{x} = 0$ *and prove that every* $\mathbf{x} \in \mathbb{R}^{3\times 1}$, *which satisfies this equation is an eigenvector of* $\mathbf{A}$.
3. *Find an orthogonal matrix* $\mathbf{Q}$ *and a diagonal matrix* $\mathbf{\Lambda}$, *such that*

$$\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{\Lambda}.$$

4. *Find the type of the surface which is described by the equation*

$$4x^2 + 7y^2 + 4z^2 - 4xy + 8xz + 4yz = 8,$$

where (x, y, z) *are the coordinates in an ordinary rectangular coordinate system in space of positive orientation. Prove also that the surface is a rotational surface and find a parametric description of the axis of rotation in the given coordinates* x, y, z.

1. By a calculation,

$$\mathbf{A}\mathbf{v} = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 7 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} = -1 \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

hence $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\lambda_1 = -1$. A normed eigenvector is

$$\mathbf{q}_1 = \frac{1}{3}(2, 1, -2).$$

2. The solution space of $\mathbf{v}^T\mathbf{x} = 0$ is spanned by the two linearly independent vectors $\mathbf{v}_2 = (1, 0, 1)$ and $\mathbf{v}_3 = (0, 2, 1)$. Then

$$\mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 7 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 8 \end{pmatrix} = 8 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\mathbf{A}\mathbf{v}_3 = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 7 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 16 \\ 8 \end{pmatrix} = 8 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix},$$

thus both $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors corresponding to $\lambda_2 = \lambda_3 = 8$.

Since $\mathbf{v}_2 + \mathbf{v}_3 = (1, 2, 2)$ also is of length 3, we may choose

$$\mathbf{q}_2 = \frac{1}{3}(1, 2, 2),$$

and since $2\mathbf{v}_2 - \mathbf{v}_3 = (2, -2, 1)$, we get

$$\mathbf{q}_3 = \frac{1}{3}(2, -2, 1).$$

3. It follows from $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ above that

$$\mathbf{Q} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Lambda} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

4. The surface corresponds precisely to the matrix $\mathbf{A}$ above, so we get by the transformation,

$$-x_1^2 + 8y_1^2 + 8z_1^2 = 8,$$

i.e. in its normalized form

$$-\left(\frac{x_1}{2\sqrt{2}}\right)^2 + y_1^2 + z_1^2 = 1.$$

This is the equation of a rotational hyperboloid of 1 sheet and with the X_1-axis as axis of rotation. In the XYZ-space the X_1-axis has the direction $\mathbf{v}$, so a parametric description is

$$t(2,1,-2), \qquad t \in \mathbb{R}.$$

Example 6.13 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 7 & 8 & 16 \\ 8 & -5 & 8 \\ 16 & 8 & 7 \end{pmatrix}.$$

1. *Prove that -9 is an eigenvalue of $\mathbf{A}$.*
2. *Given in an ordinary rectangular coordinate system in space of positive orientation a point set M of the equation*

   $$7x^2 - 5y^2 + 7z^2 + 16xy + 32xz + 16yz = 9.$$

   Find the type of M
3. *Explain that M is a rotational surface and find a directional vector of the axis of rotation.*
4. *Let α denote a plane, which contains the axis of rotation. Find the type of the curve, which is the intersection of M and α.*

1. We obtain by reduction,

   $$\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} 16 & 8 & 16 \\ 8 & 4 & 8 \\ 16 & 8 & 16 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   of rank 1, so $\lambda_1 = -9$ is an eigenvalue of multiplicity 2.

2. Two linearly independent eigenvectors are e.g. $(1,0,-1)$ and $(1,-4,1)$. They are clearly orthogonal. Furthermore, $(2,1,2)$ is an eigenvector, because it is perpendicular to a 2-dimensional eigenspace, thus it must itself lie in an eigenspace. It follows by insertion

   $$\mathbf{A}\mathbf{v}_3 = \begin{pmatrix} 7 & 8 & 16 \\ 8 & -5 & 8 \\ 16 & 8 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 54 \\ 27 \\ 54 \end{pmatrix} = 27 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

   thus $\mathbf{v}_3$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = 27$.

   Notice that

   $$\mathbf{v}_1 = (2,-2,-1) = \frac{3}{2}(1,0,-1) + \frac{1}{2}(1,-4,1),$$

   $$\mathbf{v}_2 = (1,2,-2) = \frac{3}{2}(1,0,-1) - \frac{1}{2}(1,-4,1),$$

   are other two orthogonal eigenvectors corresponding to $\lambda_1 = 9$.

Now, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are all of length 3, so an transformation matrix and a corresponding diagonal matrix are

$$\mathbf{Q} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{\Lambda} = \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 27 \end{pmatrix}.$$

By the transformation the equation is transferred into

$$-9x_1^2 - 9y_1^2 + 27z_1^2 = 9,$$

thus by reduction

$$-x_1^2 - y_1^2 + 3z_1^2 = 1.$$

3. This describes a rotational hyperboloid of 2 sheets and the Z_1-axis as axis of rotation.
   The direction of the Z_1-axis is given by the vector $(2, 1, 2)$.

4. Due to the symmetry of rotation we can choose α as the plane $y_1 = 0$. Thus we obtain the curve

   $$-x_1^2 + 3z_1^2 = 1,$$

   i.e. an hyperbola.

Example 6.14 *A bilinear function* $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *is given by*

$$g(\mathbf{x}, \mathbf{y}) = (x_1 \;\; x_2 \;\; x_3) \begin{pmatrix} 6 & 3 & 4 \\ 3 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

where $\mathbf{x} = (x_1, x_2, x_3)$ *and* $\mathbf{y} = (y_1, y_2, y_3)$*.*

1. *Prove that* g *is a scalar product in* $\mathbb{R}^3$*.*
2. *Let* $\mathbf{a} = (-1, 1, 1)$ *and* $\mathbf{b} = (1, -2, -1)$ *be given vectors of* $\mathbb{R}^3$*.*
   Find all vectors of $\mathbb{R}^3$*, which are orthogonal with respect to the scalar product* g *on both* $\mathbf{a}$ *and* $\mathbf{b}$.
3. *A map* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *is given by*
   $$f(\mathbf{x}) = g(\mathbf{x}, \mathbf{b})\mathbf{a} + g(\mathbf{x}, \mathbf{a})\mathbf{b}.$$
   Prove that f *is linear.*
4. *Find a basis of* $\ker f$*.*
5. *Find the dimension of the range and indicate a basis of the range* $f(\mathbb{R}^3)$*.*

1. The matrix is symmetric of the characteristic polynomial
   $$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 6-\lambda & 3 & 4 \\ 3 & 6-\lambda & 0 \\ 4 & 0 & 6-\lambda \end{vmatrix} = (6-\lambda)^3 - 16(6-\lambda) - 9(6-\lambda) \\ &= -(\lambda-6)\{(\lambda-6)^2 - 5^2\} = -(\lambda-6)(\lambda-1)(\lambda-11). \end{aligned}$$
   The three eigenvalues are all positive, $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = 11$, thus the matrix is positive definite, and g is an inner product.
2. Since
   $$\begin{pmatrix} 6 & 3 & 4 \\ 3 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 3 & 4 \\ 3 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -9 \\ -2 \end{pmatrix},$$
   it follows from the condition
   $$g(\mathbf{x}, \mathbf{a}) = g(\mathbf{x}, \mathbf{b}) = 0,$$
   that $\mathbf{x}$ in the ordinary system must be perpendicular to both $(1, 3, 2)$ and $(-4, -9, -2)$, i.e. it is proportional to
   $$(1,3,2) \times (4,9,2) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 3 & 2 \\ 4 & 9 & 2 \end{vmatrix} = (-12, 6, -3) = -3(4, -2, 1).$$
   The set of all vectors in $\mathbb{R}^3$, which are orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ with respect to the scalar product g, is given by
   $$\{\lambda(4, -2, 1) \mid \lambda \in \mathbb{R}\}.$$

3. Now, g is linear in its first “factor”, hence f is linear.

4. It is obvious that $(4,-2,1) \in \ker f$. If there are other vectors in $\ker f$, they must necessarily be linear combinations of $\mathbf{a}$ and $\mathbf{b}$.

   It follows by insertion of
$$\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b}$$
   that
$$f(\lambda\mathbf{a} + \mu\mathbf{b}) = \lambda g(\mathbf{a},\mathbf{b})\mathbf{a} + \mu g(\mathbf{b},\mathbf{b})\mathbf{a} + \lambda g(\mathbf{a},\mathbf{a})\mathbf{b} + \mu g(\mathbf{a},\mathbf{b})\mathbf{b}.$$
   Now,
$$\begin{aligned}
g(\mathbf{a},\mathbf{a}) &= (-1,1,1)\begin{pmatrix} 6 & 3 & 4 \\ 3 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = (-1,1,1)\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 4, \\
g(\mathbf{b},\mathbf{a}) &= (1,-2,-1)\begin{pmatrix} 6 & 3 & 4 \\ 3 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = (1,-2,1)\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = -7, \\
g(\mathbf{b},\mathbf{b}) &= (1,-2,-1)\begin{pmatrix} 6 & 3 & 4 \\ 3 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = (1,-2,-1)\begin{pmatrix} -4 \\ -9 \\ -2 \end{pmatrix} = 16,
\end{aligned}$$
   and the equation becomes
$$f(\lambda\mathbf{a} + \mu\mathbf{b}) = (-7\lambda + 16\mu)\mathbf{a} + (4\lambda - 7\mu)\mathbf{b} = \mathbf{0}.$$
   Since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, the coefficients must be 0, which again implies that $\lambda = \mu = 0$.

   We conclude that $\dim\ker f = 1$ and that $\{(4,-2,1)\}$ forms a basis.

5. The dimension of the range is $3-1=2$. It follows clearly from the structure that the set $\{\mathbf{a},\mathbf{b}\}$ must be a basis of $f(\mathbb{R}^3)$.

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